QUANTIZED ALGEBRAS OF HOLOMORPHIC FUNCTIONS ON THE POLYDISK AND ON THE BALL

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Abstract. We introduce and study noncommutative (or "quantized") versions of the algebras of holomorphic functions on the polydisk and on the ball in $\mathbb{C}^n$. Specifically, for each $q \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ we construct Fréchet algebras $\mathscr{O}_q(\mathbb{D}^n)$ and $\mathscr{O}_q(\mathbb{B}^n)$ such that for $q = 1$ they are isomorphic to the algebras of holomorphic functions on the open polydisk $\mathbb{D}^n$ and on the open ball $\mathbb{B}^n$, respectively. We show that $\mathscr{O}_q(\mathbb{D}^n)$ and $\mathscr{O}_q(\mathbb{B}^n)$ are not isomorphic provided that $|q| = 1$ and $n \geq 2$. This result can be interpreted as a q-analog of Poincaré's theorem, which asserts that $\mathbb{D}^n$ and $\mathbb{B}^n$ are not biholomorphically equivalent unless $n = 1$. In contrast, $\mathscr{O}_q(\mathbb{D}^n)$ and $\mathscr{O}_q(\mathbb{B}^n)$ are shown to be isomorphic for $|q| \neq 1$. Next we prove that $\mathscr{O}_q(\mathbb{D}^n)$ is isomorphic to a quotient of J. L. Taylor's "free polydisk algebra" (1972). This enables us to construct a Fréchet $\mathscr{O}(\mathbb{C}^\times)$-algebra $\mathscr{O}_{\mathrm{def}}(\mathbb{D}^n)$ whose "fiber" over each $q \in \mathbb{C}^\times$ is isomorphic to $\mathscr{O}_q(\mathbb{D}^n)$. Replacing the free polydisk algebra by G. Popescu's "free ball algebra" (2006), we obtain a Fréchet $\mathscr{O}(\mathbb{C}^\times)$-algebra $\mathscr{O}_{\mathrm{def}}(\mathbb{B}^n)$ with fibers isomorphic to $\mathscr{O}_q(\mathbb{B}^n)$ ($q \in \mathbb{C}^\times$). The algebras $\mathscr{O}_{\mathrm{def}}(\mathbb{D}^n)$ and $\mathscr{O}_{\mathrm{def}}(\mathbb{B}^n)$ yield continuous Fréchet algebra bundles over $\mathbb{C}^\times$ which are strict deformation quantizations (in Rieffel's sense) of $\mathscr{O}(\mathbb{D}^n)$ and $\mathscr{O}(\mathbb{B}^n)$, respectively. Finally, we study relations between our deformations and formal deformations of $\mathscr{O}(\mathbb{D}^n)$ and $\mathscr{O}(\mathbb{B}^n)$.

1. Introduction

The subject of the present paper may be roughly described as "noncommutative complex analysis", or "noncommutative complex analytic geometry". This field of mathematics is not as unified as other parts of noncommutative geometry, and there are many points of view on what noncommutative complex analysis is. The known approaches to noncommutative complex analysis differ not only in the choice of the classical objects whose noncommutative versions are constructed and studied, but also in the "degree of noncommutativity" of the new objects. According to V. Ginzburg [54], there are two types of noncommutative algebraic geometry, which may be called "noncommutative geometry in the small" and "noncommutative geometry in the large". The former is a generalization (or a deformation) of the classical geometry, and it contains the classical geometry as a special (commutative) case. In contrast, noncommutative geometry "in the large" is not a generalization of the classical theory, and it is often based on free algebra-like objects. The same terminology can equally well be applied to noncommutative complex analysis.

Perhaps the most developed approach to noncommutative complex analysis is based on operator algebra theory. This approach goes back to the foundational papers [11, 12] of W. B. Arveson, who observed that certain nonselfadjoint subalgebras of C^*-algebras sometimes behave like Banach algebras of analytic functions. A systematic development of noncommutative complex analysis "in the large" was undertaken by G. Popescu [115–129]. His point of view is motivated by the multivariable dilation theory and is based on free

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versions of the disk algebra $A(\bar{\mathbb{D}})$ and the Hardy algebra $H^\infty(\mathbb{D})$. Many related results were obtained by K. R. Davidson, D. R. Pitts, and E. G. Katsoulis [36–39], A. Arias and F. Latrémolière [7–9]. A more general version of (not necessarily free) noncommutative complex analysis based on some universal operator algebras was studied by P. Muhly and B. Solel [88–95], K. R. Davidson, C. Ramsey, and O. Shalit [40].

A common feature of the above papers is that the noncommutative algebras studied therein are always Banach (in fact, operator) algebras. A notable exception is [118], where G. Popescu introduces a Fréchet algebra which is a natural free analog of $\mathscr{O}(\mathbb{B}^n)$, the algebra of holomorphic functions on the open unit ball in $\mathbb{C}^n$. See Section 7 for more details on Popescu's algebra.

A different way of looking at noncommutative complex analysis "in the large" was suggested by J. L. Taylor [160, 161], whose main motivation was to develop spectral theory for several (not necessarily commuting) Banach space operators. In Taylor's theory, the main objects are certain Fréchet algebras containing the free algebra on finitely many generators. Ideologically, Taylor's algebras may be viewed as free analogs of the algebras of holomorphic functions on domains in $\mathbb{C}^n$. Taylor's ideas were further developed by D. Luminet [84, 85], D. Voiculescu [173, 174], D. S. Kaliuzhnyi-Verbovetskyi and V. Vinnikov [68], J. W. Helton, I. Klep, S. McCullough, and N. Slinglend [61–63], J. Agler and J. E. McCarthy [1,2]. Some parallels between Taylor's theory and the "operator algebraic" noncommutative complex analysis are discussed in [68, 94, 95]. A related approach going back to Taylor's notion of an Arens-Michael envelope [160] was developed by A. A. Dosi (Dosiev) [42–46] and the author [104–106, 109, 110].

A more algebraic view of noncommutative complex geometry is based on A. Connes' fundamental ideas [32,33]. The notion of connection introduced by Connes in [32] was used by A. Schwarz [146] to define complex structures on noncommutative tori. This line of research was further developed by M. Dieng and A. Schwarz [41] and by A. Polishchuk and A. Schwarz [112–114]. A closely related point of view was adopted by J. Rosenberg [145], M. Khalkhali, G. Landi, W. D. van Suijlekom, and A. Moatadelro [71–73], E. Beggs and S. P. Smith [13], R. Ó Buachalla [98–100]. Another approach was also initiated by Connes [33, Section VI.2], who interpreted complex structures on a compact 2-dimensional manifold M in terms of positive Hochschild cocycles on the algebra of smooth functions on M. Motivated by this, he suggested to use positivity in Hochschild cohomology as a starting point for developing noncommutative complex geometry. This point of view was developed by M. Khalkhali, G. Landi, W. D. van Suijlekom, and A. Moatadelro [loc. cit.], who found relations between complex structures on noncommutative projective spaces and twisted positive Hochschild cocycles on suitable quantized function algebras. A common feature of the above papers is that almost all concrete "noncommutative complex manifolds" that appear therein are compact. Thus section spaces of "noncommutative holomorphic bundles" over such "manifolds" are finite-dimensional, and so no functional analysis is needed for their study. We refer to [71] and [13] for a detailed discussion of this side of noncommutative complex geometry.

An enormous contribution to the development of noncommutative complex analysis "in the small" was made under the influence of L. L. Vaksman's ideas. The ambitious program initiated by Vaksman was to construct a q-analog of the function theory on bounded symmetric domains. After the publication of the pioneering paper [153] by S. Sinel'shchikov

and L. L. Vaksman, numerous results in this field were obtained by O. Bershtein, Y. Kolisnyk, D. Proskurin, A. Stolin, S. Shklyarov, S. Sinel'shchikov, L. Turowska, L. L. Vaksman, G. Zhang; see, for example, [16–21, 131, 132, 147–152, 154, 162, 164, 165, 167]. Among many interesting objects constructed by Vaksman is a q-analog of $A(\bar{\mathbb{B}}^n)$, the algebra of functions holomorphic on the open unit ball in $\mathbb{C}^n$ and continuous on its closure. In [167], Vaksman proved a q-analog of the maximum principle for functions in $A(\bar{\mathbb{B}}^n)$, and a similar result was recently obtained by Proskurin and Turowska [132] for the unit ball in the space of 2×2-matrices. For more references and further information on quantum bounded symmetric domains, see the lecture notes [166] and Vaksman's recent monograph [168].

Let us also mention the papers [143] by R. Rochberg and N. Weaver and [158] by F. H. Szafraniec, in which unbounded operators are used to investigate noncommutative analogs of the Cauchy-Riemann equations. Algebraic aspects of the Cauchy-Riemann equations over the quantum plane were studied by T. Brzeziński, H. Dąbrowski, and J. Rembieliński [31].

The book [69] by M. Kashiwara and P. Schapira suggests another look at noncommutative complex analytic geometry "in the small". The authors consider pairs $(X, \mathscr{A}_X)$, where X is a complex manifold and $\mathscr{A}_X$ is a formal deformation of the holomorphic structure sheaf $\mathscr{O}_X$ (a "DQ-algebra"). The main objects of [69] are "DQ-modules", i.e., sheaves of $\mathscr{A}_X$-modules. A number of interesting results is proved in [69], including "DQ-versions" of classical theorems by Cartan-Serre and Grauert.

Our approach to noncommutative complex analysis is slightly different. Broadly speaking, the objects we are mostly interested in are nonformal deformations of the algebras of holomorphic functions on complex Stein manifolds. There are several ways of giving an exact meaning to the phrase "nonformal deformation"; see, e.g., [14, 15, 22–28, 47, 83, 101–103, 175]. In any case, a nonformal deformation of a Fréchet algebra A should yield a family $\{A_t : t \in T\}$ of Fréchet algebras spread over a topological space T in such a way that $A_{t_0} = A$ for a fixed $t_0 \in T$. Our approach to deformations is close to that of M. A. Rieffel [135–141]; see also [74–78, 97]. Specifically, our deformations are continuous fields (or continuous bundles, see [53]) of Fréchet algebras over $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$. Moreover, each A_t is a fiber (see (A.6)) of a Fréchet $\mathscr{O}(\mathbb{C}^\times)$-algebra R, where $\mathscr{O}(\mathbb{C}^\times)$ is the "base" algebra of holomorphic functions on $\mathbb{C}^\times$. In contrast to [14, 15, 83, 103, 175], we do not require that R be topologically free over the base algebra. The reason for that will become clear in Subsection 8.2.

In the present paper, we concentrate on deformations of the algebras of holomorphic functions on two classical domains, namely the open polydisk and the open ball in $\mathbb{C}^n$. We hope that these concrete examples can serve as a basis for further research in noncommutative complex analysis "in the small". To motivate our constructions, let $q \in \mathbb{C}^\times$, and consider the algebra $\mathscr{O}_q^{\mathrm{reg}}(\mathbb{C}^n)$ of "regular functions on the quantum affine space" generated by n elements $x_1, \ldots, x_n$ subject to the relations $x_i x_j = q x_j x_i$ for all $i < j$ (see, e.g., [30]). If $q = 1$, then $\mathscr{O}_q^{\mathrm{reg}}(\mathbb{C}^n)$ is nothing but the polynomial algebra $\mathbb{C}[x_1, \ldots, x_n] = \mathscr{O}^{\mathrm{reg}}(\mathbb{C}^n)$. It is a standard fact that the monomials $x_1^{k_1} \cdots x_n^{k_n}$ $(k_1, \ldots, k_n \geq 0)$ form a basis of $\mathscr{O}_q^{\mathrm{reg}}(\mathbb{C}^n)$, so the underlying vector space of $\mathscr{O}_q^{\mathrm{reg}}(\mathbb{C}^n)$ can be identified with that of $\mathscr{O}^{\mathrm{reg}}(\mathbb{C}^n)$.

Let now $r \in (0, +\infty]$, and let $\mathbb{D}_r^n$ and $\mathbb{B}_r^n$ denote the open polydisk and the open ball of radius r in $\mathbb{C}^n$. Thus we have

$$\mathbb{D}_r^n = \Big\{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n : \max_{1 \le i \le n} |z_i| < r \Big\},$$

$$\mathbb{B}_r^n = \Big\{ z = (z_1, \ldots, z_n) \in \mathbb{C}^n : \sum_{i=1}^n |z_i|^2 < r^2 \Big\}.$$

Since $\mathscr{O}^{\mathrm{reg}}(\mathbb{C}^n)$ is dense both in $\mathscr{O}(\mathbb{D}_r^n)$ and in $\mathscr{O}(\mathbb{B}_r^n)$, it seems reasonable to define the algebras of holomorphic functions on the quantum polydisk and on the quantum ball as certain completions of $\mathscr{O}_q^{\mathrm{reg}}(\mathbb{C}^n)$. Intuitively, the idea is to "deform" the pointwise multiplication on $\mathscr{O}(\mathbb{D}_r^n)$ and $\mathscr{O}(\mathbb{B}_r^n)$ in such a way that $z_i z_j = q z_j z_i$ for all $i < j$. This idea is too naive, however, because it is not immediate whether there exists a multiplication satisfying the above condition. In fact, to give a "correct" definition of our quantized algebras, we have to "deform" not only the multiplication, but also the underlying topological vector spaces of $\mathscr{O}(\mathbb{D}_r^n)$ and $\mathscr{O}(\mathbb{B}_r^n)$ (see Remark 3.12).

The structure of the paper is as follows. After giving some preliminaries in Section 2, we proceed in Section 3 to define our quantized function algebras $\mathscr{O}_q(\mathbb{D}_r^n)$ and $\mathscr{O}_q(\mathbb{B}_r^n)$. The algebra $\mathscr{O}_q(\mathbb{D}_r^n)$ was introduced earlier in [110] in a more general multiparameter case, while the (more involved) definition of $\mathscr{O}_q(\mathbb{B}_r^n)$ is new. In the same section we show that the algebra isomorphism $\mathscr{O}_q^{\mathrm{reg}}(\mathbb{C}^n) \to \mathscr{O}_{q^{-1}}^{\mathrm{reg}}(\mathbb{C}^n)$ given by $x_i \mapsto x_{n-i}$ extends to $\mathscr{O}_q(\mathbb{D}_r^n)$ and $\mathscr{O}_q(\mathbb{B}_r^n)$. In Section 4, we show that $\mathscr{O}_q(\mathbb{D}_r^n)$ and $\mathscr{O}_q(\mathbb{B}_r^n)$ are topologically isomorphic if $|q| \ne 1$, but are not topologically isomorphic if $|q| = 1$, $n \ge 2$ and $r < \infty$. The latter result may be viewed as a q-analog of Poincaré's theorem, which asserts that $\mathbb{D}_r^n$ and $\mathbb{B}_r^n$ are not biholomorphically equivalent. In Section 5, we compare $\mathscr{O}_q(\mathbb{B}_r^n)$ (in the special case where $0 < q < 1$) with L. L. Vaksman's algebra of continuous functions on the closed quantum ball [167]. Roughly, our result is that $\mathscr{O}_q(\mathbb{B}_r^n)$ is isomorphic to the completion of $\mathscr{O}_q^{\mathrm{reg}}(\mathbb{C}^n)$ with respect to the "quantum sup-norms" over the closed balls contained in $\mathbb{B}_r^n$.

The rest of the paper is devoted to a deformation-theoretic interpretation of $\mathscr{O}_q(\mathbb{D}_r^n)$ and $\mathscr{O}_q(\mathbb{B}_r^n)$. To see where our approach comes from, let us come back again to the purely algebraic case and discuss in which sense $\mathscr{O}_q^{\mathrm{reg}}(\mathbb{C}^n)$ is a deformation (or, more exactly, a Laurent deformation) of $\mathscr{O}^{\mathrm{reg}}(\mathbb{C}^n)$. By a *Laurent deformation* of a $\mathbb{C}$-algebra A we mean a family $\{\star_q : q \in \mathbb{C}^\times\}$ of associative multiplications on A such that $\star_1$ is the initial multiplication on A and such that for every $a, b \in A$ the function $q \in \mathbb{C}^\times \mapsto a \star_q b \in A$ is an A-valued Laurent polynomial. Equivalently, a Laurent deformation of A is a $\mathbb{C}[t^{\pm 1}]$-algebra R together with an algebra isomorphism $R/(t-1)R \cong A$ such that R is a free $\mathbb{C}[t^{\pm 1}]$-module. To see that the above definitions are equivalent, observe that for each $q \in \mathbb{C}^\times$ we have a vector space isomorphism $R/(t-q)R \cong A$, so we can let $(A, \star_q) = R/(t-q)R$. If we identify the underlying vector spaces of $\mathscr{O}_q^{\mathrm{reg}}(\mathbb{C}^n)$ and $\mathscr{O}^{\mathrm{reg}}(\mathbb{C}^n)$ via the isomorphism $x^k \mapsto x^k$ ($k \in \mathbb{Z}_+^n$), then the resulting family of multiplications on $\mathscr{O}^{\mathrm{reg}}(\mathbb{C}^n)$ clearly becomes a Laurent deformation of $\mathscr{O}^{\mathrm{reg}}(\mathbb{C}^n)$.

The above definition of a Laurent deformation has a natural "holomorphic" analog in the case where A is a complete locally convex topological algebra. Specifically, we can replace $\mathbb{C}[t^{\pm 1}]$ by the algebra $\mathscr{O}(\mathbb{C}^\times)$ of holomorphic functions on $\mathbb{C}^\times$, and consider "topologically free" modules instead of free modules (cf. also Subsection 8.2). This approach is systematically developed in [103], but it is too restrictive for our purposes. The reason was already mentioned above; in general, the multiplication on $\mathscr{O}(\mathbb{D}_r^n)$ cannot

be deformed in such a way that $z_i z_j = q z_j z_i$ for all $i < j$. Therefore there is no chance to construct a holomorphic deformation of $\mathscr{O}(\mathbb{D}_r^n)$ in the sense of [103].

Our approach is based on the following observation. Let R denote the Laurent deformation of $\mathscr{O}^{\mathrm{reg}}(\mathbb{C}^n)$ introduced above. Thus R is a $\mathbb{C}[t^{\pm 1}]$-algebra such that for each $q \in \mathbb{C}^\times$ we have $R/(t-q)R \cong \mathscr{O}_q^{\mathrm{reg}}(\mathbb{C}^n)$ and such that R is free over $\mathbb{C}[t^{\pm 1}]$. It is easy to see that we have a $\mathbb{C}[t^{\pm 1}]$-algebra isomorphism

$$R \cong (\mathbb{C}[t^{\pm 1}] \otimes F_n)/I, \tag{1.1}$$

where $F_n = \mathbb{C}\langle \zeta_1, \ldots, \zeta_n \rangle$ is the free algebra and I is the two-sided ideal of $\mathbb{C}[t^{\pm 1}] \otimes F_n$ generated by $\zeta_i \zeta_j - t \zeta_j \zeta_i$ $(i < j)$; cf. [56]. To construct a deformation of $\mathscr{O}(D)$ (where D is either $\mathbb{D}_r^n$ or $\mathbb{B}_r^n$), we suggest to replace $\mathbb{C}[t^{\pm 1}]$ by $\mathscr{O}(\mathbb{C}^\times)$ in (1.1), to replace $\otimes$ by $\widehat{\otimes}$ (where $\widehat{\otimes}$ is the completed projective tensor product), and finally, to replace F_n by a suitable "analytic free algebra" (i.e., the completion of F_n with respect to a suitable locally convex topology).

Of course, the nontrivial part of the above program is to construct an appropriate completion of F_n. If $D = \mathbb{D}_r^n$, then we have at least two natural candidates for such a completion, namely the "free polydisk" algebras $\mathscr{F}^{\mathrm{T}}(\mathbb{D}_r^n)$ and $\mathscr{F}(\mathbb{D}_r^n)$ introduced in [160] and [110], respectively. In [110], we proved that the quotient of $\mathscr{F}(\mathbb{D}_r^n)$ modulo the closed two-sided ideal generated by $\zeta_i \zeta_j - q \zeta_j \zeta_i$ $(i < j)$ is topologically isomorphic to $\mathscr{O}_q(\mathbb{D}_r^n)$. In Section 6 of the present paper, we show that a similar result holds for $\mathscr{F}^{\mathrm{T}}(\mathbb{D}_r^n)$. To this end, we prove that $\mathscr{F}^{\mathrm{T}}(\mathbb{D}_r^n)$ has a remarkable universal property formulated in terms of the joint spectral radius.

To perform a similar construction in the case where $D = \mathbb{B}_r^n$, we discuss in Section 7 the "free ball" algebra $\mathscr{F}(\mathbb{B}_r^n)$ introduced by G. Popescu in [118]. We give an alternative definition of $\mathscr{F}(\mathbb{B}_r^n)$, which seems to be more appropriate for our purposes, show that Popescu's definition is equivalent to ours, and prove that the quotient of $\mathscr{F}(\mathbb{B}_r^n)$ modulo the closed two-sided ideal generated by $\zeta_i \zeta_j - q \zeta_j \zeta_i$ $(i < j)$ is topologically isomorphic to $\mathscr{O}_q(\mathbb{B}_r^n)$. We believe that the interpretations of $\mathscr{O}_q(\mathbb{D}_r^n)$ and $\mathscr{O}_q(\mathbb{B}_r^n)$ as quotients of $\mathscr{F}(\mathbb{D}_r^n)$ and $\mathscr{F}(\mathbb{B}_r^n)$, respectively, indicate that our *ad hoc* definitions of $\mathscr{O}_q(\mathbb{D}_r^n)$ and $\mathscr{O}_q(\mathbb{B}_r^n)$ given in Section 3 are indeed the "correct" ones.

The results of Sections 6 and 7 are then applied in Section 8 to construct Fréchet $\mathscr{O}(\mathbb{C}^\times)$-algebras which can be viewed as "holomorphic deformations" of $\mathscr{O}(\mathbb{D}_r^n)$ and $\mathscr{O}(\mathbb{B}_r^n)$. Specifically, using (1.1) as a motivation, for every $F \in \{\mathscr{F}(\mathbb{D}_r^n), \mathscr{F}^{\mathrm{T}}(\mathbb{D}_r^n), \mathscr{F}(\mathbb{B}_r^n)\}$ we consider the quotient $(\mathscr{O}(\mathbb{C}^\times) \mathbin{\widehat{\otimes}} F)/I$, where I is the closed two-sided ideal of $\mathscr{O}(\mathbb{C}^\times) \mathbin{\widehat{\otimes}} F$ generated by $\zeta_i \zeta_j - t \zeta_j \zeta_i$ $(i < j)$. This yields three Fréchet $\mathscr{O}(\mathbb{C}^\times)$-algebras denoted by $\mathscr{O}_{\mathrm{def}}(\mathbb{D}_r^n)$, $\mathscr{O}_{\mathrm{def}}^{\mathrm{T}}(\mathbb{D}_r^n)$, and $\mathscr{O}_{\mathrm{def}}(\mathbb{B}_r^n)$, respectively. If we let $R = \mathscr{O}_{\mathrm{def}}(\mathbb{D}_r^n)$ or $R = \mathscr{O}_{\mathrm{def}}^{\mathrm{T}}(\mathbb{D}_r^n)$, then for every $q \in \mathbb{C}^\times$ we have $R/\overline{(t-q)R} \cong \mathscr{O}_q(\mathbb{D}_r^n)$. Similarly, if we let $R = \mathscr{O}_{\mathrm{def}}(\mathbb{B}_r^n)$, then for every $q \in \mathbb{C}^\times$ we have $R/\overline{(t-q)R} \cong \mathscr{O}_q(\mathbb{B}_r^n)$. Moreover, we show in Subsection 8.1 that the $\mathscr{O}(\mathbb{C}^\times)$-algebras $\mathscr{O}_{\mathrm{def}}(\mathbb{D}_r^n)$ and $\mathscr{O}_{\mathrm{def}}^{\mathrm{T}}(\mathbb{D}_r^n)$ are isomorphic, in spite of the fact that $\mathscr{F}(\mathbb{D}_r^n)$ and $\mathscr{F}^{\mathrm{T}}(\mathbb{D}_r^n)$ are not isomorphic in general. In Subsection 8.2, we prove that $\mathscr{O}_{\mathrm{def}}(\mathbb{D}_r^n)$ is not topologically free over $\mathscr{O}(\mathbb{C}^\times)$ (in contrast to its algebraic prototype (1.1), which is free over $\mathbb{C}[t^{\pm 1}]$). In Subsections 8.3 and 8.4, we show that the Fréchet algebra bundles $\mathsf{E}(\mathbb{D}_r^n)$ and $\mathsf{E}(\mathbb{B}_r^n)$ associated to $\mathscr{O}_{\mathrm{def}}(\mathbb{D}_r^n)$ and $\mathscr{O}_{\mathrm{def}}(\mathbb{B}_r^n)$ are continuous, and that they form strict Fréchet deformation quantizations of $\mathscr{O}(\mathbb{D}_r^n)$ and $\mathscr{O}(\mathbb{B}_r^n)$ in the sense of Rieffel. In Subsection 8.5, we establish a relation between deformations in our sense and formal deformations. Specifically, we show that the $\mathbb{C}[[h]]$-algebras $\mathbb{C}[[h]] \mathbin{\widehat{\otimes}}_{\mathscr{O}(\mathbb{C}^\times)} \mathscr{O}_{\mathrm{def}}(\mathbb{D}_r^n)$

and $\mathbb{C}[[h]]\,\widehat{\otimes}_{\mathscr{O}(\mathbb{C}^\times)}\,\mathscr{O}_{\mathrm{def}}(\mathbb{B}^n_r)$ are formal deformations of $\mathscr{O}(\mathbb{D}^n_r)$ and $\mathscr{O}(\mathbb{B}^n_r)$, respectively. The nontrivial point here is to show that the above algebras are topologically free over $\mathbb{C}[[h]]$. Finally, Appendix A contains some auxiliary facts on bundles of locally convex spaces and algebras.

Some of the results of the present paper were announced in [108].

2. Preliminaries and notation

We shall work over the field $\mathbb{C}$ of complex numbers. All algebras are assumed to be associative and unital, and all algebra homomorphisms are assumed to be unital (i.e., to preserve identity elements). By a *Fréchet algebra* we mean a complete metrizable locally convex algebra (i.e., a topological algebra whose underlying space is a Fréchet space). A *locally m-convex algebra* [87] is a topological algebra A whose topology can be defined by a family of submultiplicative seminorms (i.e., seminorms $\|\cdot\|$ satisfying $\|ab\| \le \|a\|\|b\|$ for all $a, b \in A$). A complete locally m-convex algebra is called an *Arens-Michael algebra* [60].

Throughout we will use the following multi-index notation. Let $\mathbb{Z}_+ = \mathbb{N}\cup\{0\}$ denote the set of all nonnegative integers. For each $n \in \mathbb{N}$ and each $d \in \mathbb{Z}_+$, let $W_{n,d} = \{1, \ldots, n\}^d$, and let $W_n = \bigsqcup_{d\in\mathbb{Z}_+} W_{n,d}$. Thus a typical element of W_n is a d-tuple $\alpha = (\alpha_1, \ldots, \alpha_d)$ of arbitrary length $d \in \mathbb{Z}_+$, where $\alpha_j \in \{1, \ldots, n\}$ for all j. The only element of $W_{n,0}$ will be denoted by $*$. For each $\alpha \in W_{n,d} \subset W_n$, let $|\alpha| = d$. Given an algebra A, an n-tuple $a = (a_1, \ldots, a_n) \in A^n$, and $\alpha = (\alpha_1, \ldots, \alpha_d) \in W_n$, we let $a_\alpha = a_{\alpha_1} \cdots a_{\alpha_d} \in A$ if $d > 0$; it is also convenient to set $a_* = 1 \in A$. Given $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n_+$, we let $a^k = a_1^{k_1} \cdots a_n^{k_n}$. If the a_i's are invertible, then a^k makes sense for all $k \in \mathbb{Z}^n$. As usual, for each $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$, we let $|k| = |k_1| + \cdots + |k_n|$. Given $d \in \mathbb{Z}_+$, we let

$$(\mathbb{Z}^n_+)_d = \{k \in \mathbb{Z}^n_+ : |k| = d\}.$$

We will also use the standard notation related to q-numbers (see, e.g., [52, 67, 70]). Given $q \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ and $k \in \mathbb{N}$, let

$$[k]_q = 1 + q + \cdots + q^{k-1}; \quad [k]_q! = [1]_q[2]_q \cdots [k]_q.$$

It is also convenient to let $[0]_q! = 1$. If $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n_+$, then we let $[k]_q! = [k_1]_q! \cdots [k_n]_q!$. If $|q| < 1$, then for each $a \in \mathbb{C}$ we let $(a; q)_\infty = \prod_{j=0}^{\infty}(1 - aq^j)$.

Given an algebra A, the *Arens-Michael envelope* of A is the completion of A with respect to the family of all submultiplicative seminorms on A. Equivalently, the Arens-Michael envelope of A is an Arens-Michael algebra $\widehat{A}$ together with a natural isomorphism

$$\mathrm{Hom}_{\mathsf{Alg}}(A, B) \cong \mathrm{Hom}_{\mathsf{AM}}(\widehat{A}, B) \qquad (B \in \mathsf{AM}), \tag{2.1}$$

where Alg is the category of algebras and algebra homomorphisms, and AM is the category of Arens-Michael algebras and continuous algebra homomorphisms. Moreover, the correspondence $A \mapsto \widehat{A}$ is a functor from Alg to AM, and this functor is left adjoint to the forgetful functor $\mathsf{AM} \to \mathsf{Alg}$ (see (2.1)).

Arens-Michael envelopes were introduced by J. L. Taylor [159] under the name of "completed locally m-convex envelopes". Now it is customary to call them "Arens-Michael envelopes", following the terminology suggested by A. Ya. Helemskii [60].

Here are some basic examples of Arens-Michael envelopes. If $A = \mathbb{C}[z_1, \ldots, z_n]$ is the polynomial algebra, then the Arens-Michael envelope of A is the algebra $\mathscr{O}(\mathbb{C}^n)$ of entire functions on $\mathbb{C}^n$ [160]. More generally [106, Example 3.6], if X is an affine algebraic variety and $A = \mathscr{O}^{\mathrm{reg}}(X)$ is the algebra of regular functions on X, then $\widehat{A}$ is the algebra

$\mathscr{O}(X)$ of holomorphic functions on X. A similar result holds in the case where $(X, \mathscr{O}_X^{\mathrm{reg}})$ is an affine scheme of finite type over $\mathbb{C}$ [110, Example 7.2].

Let $q \in \mathbb{C}^\times$. Recall from Section 1 that the algebra $\mathscr{O}_q^{\mathrm{reg}}(\mathbb{C}^n)$ *of regular functions on the quantum affine n-space* is generated by n elements $x_1, \ldots, x_n$ subject to the relations $x_i x_j = q x_j x_i$ for all $i < j$ (see, e.g., [30]). The Arens-Michael envelope of $\mathscr{O}_q^{\mathrm{reg}}(\mathbb{C}^n)$ is denoted by $\mathscr{O}_q(\mathbb{C}^n)$ and is called the *algebra of holomorphic functions on the quantum affine space* [106]. As was shown in [106, Theorem 5.11], we have

$$\mathscr{O}_q(\mathbb{C}^n) = \Big\{ a = \sum_{k \in \mathbb{Z}_+^n} c_k x^k : \|a\|_\rho = \sum_{k \in \mathbb{Z}_+^n} |c_k| w_q(k) \rho^{|k|} < \infty \ \forall \rho > 0 \Big\}, \tag{2.2}$$

where

$$w_q(k) = \begin{cases} 1 & \text{if } |q| \geq 1; \\ |q|^{\sum_{i<j} k_i k_j} & \text{if } |q| < 1. \end{cases} \tag{2.3}$$

The topology on $\mathscr{O}_q(\mathbb{C}^n)$ is given by the norms $\|\cdot\|_\rho$ ($\rho > 0$), and the multiplication is uniquely determined by $x_i x_j = q x_j x_i$ ($i < j$). Moreover, each norm $\|\cdot\|_\rho$ is submultiplicative.

Let now $F_n = \mathbb{C}\langle \zeta_1, \ldots, \zeta_n \rangle$ be the free algebra. As was observed by Taylor [160], the Arens-Michael envelope of F_n is isomorphic to

$$\mathscr{F}_n = \Big\{ a = \sum_{\alpha \in W_n} c_\alpha \zeta_\alpha : \|a\|_\rho = \sum_{\alpha \in W_n} |c_\alpha| \rho^{|\alpha|} < \infty \ \forall \rho > 0 \Big\}. \tag{2.4}$$

The topology on $\mathscr{F}_n$ is given by the norms $\|\cdot\|_\rho$ ($\rho > 0$), and the multiplication is given by concatenation (like on F_n). Moreover, each norm $\|\cdot\|_\rho$ is submultiplicative.

We refer to [44, 46, 106] for explicit descriptions of Arens-Michael envelopes of some other finitely generated algebras, including quantum tori, quantum Weyl algebras, the algebra of quantum 2×2-matrices, and universal enveloping algebras. Further results on Arens-Michael envelopes can be found in [42, 43, 45, 104, 105].

Let us now discuss some basic facts on vector-valued power series spaces. The material given below is fairly standard, but we have not found a convenient reference, and that is why we give full details.

Let $n \in \mathbb{N}$, and let $\mathbb{E} = \{E_k : k \in \mathbb{Z}_+^n\}$ be a family of Banach spaces. For each $p \geq 1$ and each $r \in (0, +\infty]$, we let

$$\Lambda_r^p(\mathbb{E}) = \Big\{ x = \sum_{k \in \mathbb{Z}_+^n} x_k : \|x\|_\rho^{(p)} = \Big(\sum_{k \in \mathbb{Z}_+^n} \|x_k\|^p \rho^{p|k|} \Big)^{1/p} < \infty \quad \forall \rho \in (0, r) \Big\},$$

$$\Lambda_r^\infty(\mathbb{E}) = \Big\{ x = \sum_{k \in \mathbb{Z}_+^n} x_k : \|x\|_\rho^{(\infty)} = \sup_{k \in \mathbb{Z}_+^n} \|x_k\| \rho^{|k|} < \infty \quad \forall \rho \in (0, r) \Big\}.$$

By Minkowski's inequality, for each $p \in [1, +\infty]$, $\Lambda_r^p(\mathbb{E})$ is a vector subspace of $\prod_{k \in \mathbb{Z}_+} E_k$, and for each $\rho \in (0, r)$, $\|\cdot\|_\rho^{(p)}$ is a norm on $\Lambda_r^p(\mathbb{E})$. We endow $\Lambda_r^p(\mathbb{E})$ with the locally convex topology determined by the family $\{\|\cdot\|_\rho^{(p)} : \rho \in (0, r)\}$ of norms.

Lemma 2.1. *For each $p, q \in [1, +\infty]$, $\Lambda_r^p(\mathbb{E}) = \Lambda_r^q(\mathbb{E})$ as locally convex spaces. Moreover, if $p < q$ and $0 < \rho < \tau < r$, then we have*

$$\|\cdot\|_\rho^{(q)} \leq \|\cdot\|_\rho^{(p)} \leq \left(\frac{\tau^\ell}{\tau^\ell - \rho^\ell} \right)^{\frac{n}{\ell}} \|\cdot\|_\tau^{(q)} \tag{2.5}$$

on $\Lambda_r^p(\mathbb{E})$, where

$$\ell = \left(\frac{1}{p} - \frac{1}{q}\right)^{-1}.$$

Finally, $\Lambda_r^p(\mathbb{E})$ is a Fréchet space.

Proof. The metrizability of $\Lambda_r^p(\mathbb{E})$ is immediate from the fact that the family $\{\|\cdot\|_\rho^{(p)} : \rho \in (0,r)\}$ of norms is equivalent to the countable subfamily $\{\|\cdot\|_{\rho_i}^{(p)} : i \in \mathbb{N}\}$, where (ρ_i) is any increasing sequence converging to r. A standard argument (see, e.g., [86, Lemma 27.1]) shows that $\Lambda_r^p(\mathbb{E})$ is complete.

Recall that we have $\ell^p \subset \ell^q$ whenever $p < q$, and that the ℓ^q-norm of any $a \in \ell^p$ is less than or equal to the ℓ^p-norm of a. Therefore $\Lambda_r^p(\mathbb{E}) \subseteq \Lambda_r^q(\mathbb{E})$, and, for each $\rho \in (0,r)$, we have $\|\cdot\|_\rho^{(q)} \le \|\cdot\|_\rho^{(p)}$ on $\Lambda^p(\mathbb{E})$. Assume now that $q < \infty$, and let

$$s = \frac{q}{p}, \quad s' = \frac{q}{q-p}, \quad \text{so that} \quad \frac{1}{s} + \frac{1}{s'} = 1 \quad \text{and} \quad ps' = \ell.$$

By using Hölder's inequality, for each $x \in \Lambda_r^q(\mathbb{E})$ and each $\tau \in (\rho, r)$ we obtain

$$\begin{aligned}
\|x\|_\rho^{(p)} &= \left(\sum_{k\in\mathbb{Z}_+^n} \|x_k\|^p \tau^{p|k|} \left(\frac{\rho}{\tau}\right)^{p|k|}\right)^{1/p} \\
&\le \left(\sum_{k\in\mathbb{Z}_+^n} \|x_k\|^{ps} \tau^{ps|k|}\right)^{1/ps} \left(\sum_{k\in\mathbb{Z}_+^n} \left(\frac{\rho}{\tau}\right)^{ps'|k|}\right)^{1/ps'} \\
&= \|x\|_\tau^{(q)} \left(\sum_{m=0}^\infty \left(\frac{\rho}{\tau}\right)^{\ell m}\right)^{n/\ell} = \left(\frac{\tau^\ell}{\tau^\ell - \rho^\ell}\right)^{n/\ell} \|x\|_\tau^{(q)}.
\end{aligned}$$

Thus $\Lambda_r^q(\mathbb{E}) \subseteq \Lambda_r^p(\mathbb{E})$, and (2.5) holds. This completes the proof in the case where $q < \infty$. The case $q = \infty$ is similar. $\square$

3. Quantum polydisk and quantum ball

Let us start by recalling a well-known power series characterization of the algebra $\mathscr{O}(\mathbb{D}_r^n)$ of holomorphic functions on the polydisk (see, e.g., [86, Example 27.27]). We have

$$\mathscr{O}(\mathbb{D}_r^n) \cong \left\{a = \sum_{k\in\mathbb{Z}_+^n} c_k z^k : \|a\|_{\mathbb{D},\rho} = \sum_{k\in\mathbb{Z}_+^n} |c_k| \rho^{|k|} < \infty \ \forall \rho \in (0,r)\right\}. \tag{3.1}$$

The space on the right-hand side of (3.1) is a subalgebra of $\mathbb{C}[[z_1, \ldots, z_n]]$ and a Fréchet-Arens-Michael algebra under the family $\{\|\cdot\|_{\mathbb{D},\rho} : \rho \in (0,r)\}$ of submultiplicative norms. The Fréchet algebra isomorphism (3.1) takes each holomorphic function on $\mathbb{D}_r^n$ to its Taylor expansion at 0.

The following definition is motivated by (3.1) and (2.2).

Definition 3.1 ([109,110]). Let $q \in \mathbb{C}^\times$, and let $r \in (0, +\infty]$. The *algebra of holomorphic functions on the quantum n-polydisk of radius r* is

$$\mathscr{O}_q(\mathbb{D}_r^n) = \left\{a = \sum_{k\in\mathbb{Z}_+^n} c_k x^k : \|a\|_{\mathbb{D},\rho} = \sum_{k\in\mathbb{Z}_+^n} |c_k| w_q(k) \rho^{|k|} < \infty \ \forall \rho \in (0,r)\right\},$$

where the function $w_q\colon \mathbb{Z}_+^n \to \mathbb{R}_+$ is given by (2.3). The topology on $\mathscr{O}_q(\mathbb{D}_r^n)$ is given by the norms $\|\cdot\|_{\mathbb{D},\rho}$ $(0<\rho<r)$, and the multiplication on $\mathscr{O}_q(\mathbb{D}_r^n)$ is uniquely determined by $x_i x_j = q x_j x_i$ for all $i<j$.

In other words, $\mathscr{O}_q(\mathbb{D}_r^n)$ is the completion of $\mathscr{O}_q^{\mathrm{reg}}(\mathbb{C}^n)$ with respect to the family $\{\|\cdot\|_{\mathbb{D},\rho} : \rho\in(0,r)\}$ of submultiplicative norms. Letting $q=1$ in Definition 3.1 and comparing with (3.1), we see that the map $\mathscr{O}(\mathbb{D}_r^n)\to\mathscr{O}_1(\mathbb{D}_r^n)$ taking each $f\in\mathscr{O}(\mathbb{D}_r^n)$ to its Taylor series at 0 is a Fréchet algebra isomorphism. Note that, if $r=\infty$, then $\mathscr{O}_q(\mathbb{D}_r^n)=\mathscr{O}_q(\mathbb{C}^n)$ (see (2.2)).

To define the algebra of holomorphic functions on the quantum ball, we need the following generalization of (3.1) due to L. A. Aizenberg and B. S. Mityagin [3] (see also [144]). Given a complete bounded Reinhardt domain $D\subset\mathbb{C}^n$, let

$$b_k(D)=\sup_{z\in D}|z^k|=\sup_{z\in\partial D}|z^k| \qquad (k\in\mathbb{Z}_+^n).$$

Aizenberg and Mityagin proved that there is a topological isomorphism

$$\mathscr{O}(D)\cong\Big\{f=\sum_{k\in\mathbb{Z}_+^n}c_k z^k : \|f\|_s=\sum_{k\in\mathbb{Z}_+^n}|c_k|b_k(D)s^{|k|}<\infty\ \forall s\in(0,1)\Big\} \tag{3.2}$$

taking each holomorphic function on D to its Taylor series at 0.

We clearly have $b_k(\mathbb{D}_r^n)=r^{|k|}$, so (3.1) is a special case of (3.2). Consider now the case $D=\mathbb{B}_r^n$.

Lemma 3.2. *For each $r\in(0,+\infty)$, we have*

$$b_k(\mathbb{B}_r^n)=\left(\frac{k^k}{|k|^{|k|}}\right)^{\frac12} r^{|k|}.$$

Proof. This is an elementary calculation involving Lagrange multipliers. ☐

Corollary 3.3. *For each $r\in(0,+\infty]$, there is a topological isomorphism*

$$\mathscr{O}(\mathbb{B}_r^n)\cong\left\{f=\sum_{k\in\mathbb{Z}_+^n}c_k z^k : \|f\|'_{\mathbb{B},\rho}=\sum_{k\in\mathbb{Z}_+^n}|c_k|\left(\frac{k^k}{|k|^{|k|}}\right)^{\frac12}\rho^{|k|}<\infty\ \forall\rho\in(0,r)\right\} \tag{3.3}$$

taking each holomorphic function on $\mathbb{B}_r^n$ to its Taylor series at 0.

Proof. Let $\mathscr{O}'(\mathbb{B}_r^n)$ denote the power series space on the right-hand side of (3.3). If $r<\infty$, then the isomorphism $\mathscr{O}(\mathbb{B}_r^n)\cong\mathscr{O}'(\mathbb{B}_r^n)$ is immediate from (3.2) and Lemma 3.2. To prove the result for $r=\infty$, observe that we have obvious topological isomorphisms

$$\mathscr{O}(\mathbb{C}^n)\cong\varprojlim_{r<\infty}\mathscr{O}(\mathbb{B}_r^n),\quad \mathscr{O}'(\mathbb{C}^n)\cong\varprojlim_{r<\infty}\mathscr{O}'(\mathbb{B}_r^n). \tag{3.4}$$

Moreover, the isomorphisms $\varphi_r\colon\mathscr{O}(\mathbb{B}_r^n)\to\mathscr{O}'(\mathbb{B}_r^n)$ (see (3.3)), which are already defined for all $r<\infty$, are clearly compatible with the linking maps of the inverse systems (3.4). Letting $\varphi_\infty=\varprojlim_{r<\infty}\varphi_r$, we obtain the required topological isomorphism for $r=\infty$. ☐

For our purposes, a slightly different power series representation of $\mathscr{O}(\mathbb{B}_r^n)$ is needed. Let us start with an elementary lemma.

Lemma 3.4. *For each $k\in\mathbb{Z}_+^n$, let*

$$a_k=\frac{k!}{|k|!},\quad b_k=\frac{k^k}{|k|^{|k|}}. \tag{3.5}$$

Then

$$\lim_{k\to\infty}\left(\frac{a_k}{b_k}\right)^{\frac{1}{|k|}}=1. \tag{3.6}$$

Proof. For each $m\in\mathbb{Z}_+$, let $m^+=m+1$. By Stirling's formula, there exists a function $\theta\colon\mathbb{Z}_+\to\mathbb{R}$ such that $\theta(m)\to 0$ as $m\to\infty$ and[1]

$$m!=\sqrt{2\pi m^+}\,m^m e^{-m+\theta(m)} \qquad (m\in\mathbb{Z}_+). \tag{3.7}$$

Therefore for each $k=(k_1,\dots,k_n)\in\mathbb{Z}_+^n$ we have

$$\begin{aligned} k! &= (2\pi)^{n/2}(k_1^+\cdots k_n^+)^{1/2}k^k e^{-|k|+\sum_i\theta(k_i)},\\ |k|! &= (2\pi)^{1/2}(|k|+1)^{1/2}|k|^{|k|}e^{-|k|+\theta(|k|)}, \end{aligned}$$

whence

$$\frac{a_k}{b_k}=(2\pi)^{\frac{n-1}{2}}\left(\frac{k_1^+\cdots k_n^+}{|k|+1}\right)^{\frac12}e^{\tau(k)} \qquad (k\in\mathbb{Z}_+^n),$$

where $\tau(k)=\sum_i\theta(k_i)-\theta(|k|)$ is bounded. Now, in order to prove (3.6), it remains to show that

$$\lim_{k\to\infty}\left(\frac{k_1^+\cdots k_n^+}{|k|+1}\right)^{\frac{1}{2|k|}}=1. \tag{3.8}$$

We have

$$\begin{aligned} \left(\frac{k_1^+\cdots k_n^+}{|k|+1}\right)^{\frac{1}{2|k|}} &\le \left(\frac{(|k|+1)^n}{|k|+1}\right)^{\frac{1}{2|k|}}=(|k|+1)^{\frac{n-1}{2|k|}}\to 1 \qquad (k\to\infty);\\ \left(\frac{k_1^+\cdots k_n^+}{|k|+1}\right)^{\frac{1}{2|k|}} &\ge \left(\frac{\max_i k_i^+}{|k|+1}\right)^{\frac{1}{2|k|}}\ge\left(\frac{1}{n}\right)^{\frac{1}{2|k|}}\to 1 \qquad (k\to\infty). \end{aligned}$$

This proves (3.8), which, in turn, implies (3.6). ☐

Proposition 3.5. *For each $r\in(0,+\infty]$, there is a topological isomorphism*

$$\mathscr{O}(\mathbb{B}_r^n)\cong\left\{f=\sum_{k\in\mathbb{Z}_+^n}c_kz^k : \|f\|_{\mathbb{B},\rho}=\sum_{k\in\mathbb{Z}_+^n}|c_k|\left(\frac{k!}{|k|!}\right)^{\frac12}\rho^{|k|}<\infty\ \forall\rho\in(0,r)\right\} \tag{3.9}$$

taking each holomorphic function on $\mathbb{B}_r^n$ to its Taylor series at 0.

Proof. In view of Corollary 3.3, it suffices to show that the families

$$\{\|\cdot\|_{\mathbb{B},\rho}:\rho\in(0,r)\} \quad\text{and}\quad \{\|\cdot\|'_{\mathbb{B},\rho}:\rho\in(0,r)\} \tag{3.10}$$

of norms are equivalent on $\mathbb{C}[z_1,\dots,z_n]$. Define sequences $(a_k)_{k\in\mathbb{Z}_+^n}$ and $(b_k)_{k\in\mathbb{Z}_+^n}$ by (3.5). Fix any $\rho\in(0,r)$, and choose $\rho_1\in(\rho,r)$. By Lemma 3.4, there exists $K\in\mathbb{N}$ such that

$$\frac{\rho}{\rho_1}\le\left(\frac{a_k}{b_k}\right)^{\frac{1}{2|k|}}\le\frac{\rho_1}{\rho} \qquad (|k|\ge K). \tag{3.11}$$

Using (3.11), we can find $C>0$ such that

$$a_k^{1/2}\le Cb_k^{1/2}\left(\frac{\rho_1}{\rho}\right)^{|k|},\quad b_k^{1/2}\le Ca_k^{1/2}\left(\frac{\rho_1}{\rho}\right)^{|k|} \qquad (k\in\mathbb{Z}_+^n).$$

[1] We use m^+ instead of m in (3.7) in order to cover the case $m=0$, which is essential when we pass to multi-indices.

Hence for each $f=\sum_k c_k z^k \in \mathbb{C}[z_1,\ldots,z_n]$ we have

$$\|f\|_{\mathbb{B},\rho}=\sum_k |c_k| a_k^{1/2}\rho^{|k|} \le C\sum_k |c_k| b_k^{1/2}\rho_1^{|k|} = C\|f\|'_{\mathbb{B},\rho_1},$$
$$\|f\|'_{\mathbb{B},\rho}=\sum_k |c_k| b_k^{1/2}\rho^{|k|} \le C\sum_k |c_k| a_k^{1/2}\rho_1^{|k|} = C\|f\|_{\mathbb{B},\rho_1}.$$

Thus the families (3.10) of norms on $\mathbb{C}[z_1,\ldots,z_n]$ are equivalent, and hence the power series spaces on the right-hand sides of (3.9) and (3.3) coincide. This completes the proof. $\square$

Remark 3.6. At the moment, it is not obvious whether the norms $\|\cdot\|_{\mathbb{B},\rho}$ defined by (3.9) are submultiplicative on $\mathscr{O}(\mathbb{B}^n_r)$. In fact they are, and this can be proved directly by using the inequality

$$\binom{m}{n}\binom{p}{q} \le \binom{m+p}{n+q},$$

which is immediate from the Chu-Vandermonde formula (see, e.g., [157, 1.1.17]). We omit the details, because a more general result will be proved in Theorem 3.9.

Now, in order to define a q-analog of $\mathscr{O}(\mathbb{B}^n_r)$, we need to "quantize" the norms $\|\cdot\|_{\mathbb{B},\rho}$ given by (3.9). This will be done in the following two lemmas.

Lemma 3.7. *For each $q>0$ and for each $k,\ell\in\mathbb{Z}^n_+$, we have*

$$\frac{\big[|k+\ell|\big]_q!}{[k+\ell]_q!} \ge \frac{\big[|k|\big]_q!}{[k]_q!}\,\frac{\big[|\ell|\big]_q!}{[\ell]_q!}\, q^{\sum_{i<j} k_i\ell_j}. \tag{3.12}$$

Proof. We use induction on n. For $n=2$, the q-analog of the Chu-Vandermonde formula (see, e.g., [79, 2.1.2, Proposition 3]) implies that

$$\begin{aligned}\binom{k_1+\ell_1+k_2+\ell_2}{k_1+\ell_1}_q &= \sum_{j=0}^{k_1+k_2}\binom{k_1+k_2}{j}_q\binom{\ell_1+\ell_2}{k_1+\ell_1-j}_q q^{j(\ell_2-k_1+j)}\\ &\ge \binom{k_1+k_2}{k_1}_q\binom{\ell_1+\ell_2}{\ell_1}_q q^{k_1\ell_2}.\end{aligned}$$

This is exactly (3.12) for $n=2$. Suppose now that $n\ge 3$, and, for each $k=(k_1,\ldots,k_n)\in\mathbb{Z}^n_+$, let $k'=(k_1,\ldots,k_{n-1})\in\mathbb{Z}^{n-1}_+$. By the induction hypothesis, we have

$$\frac{\big[|k'+\ell'|\big]_q!}{[k'+\ell']_q!} \ge \frac{\big[|k'|\big]_q!}{[k']_q!}\,\frac{\big[|\ell'|\big]_q!}{[\ell']_q!}\, q^{\sum_{i<j\le n-1} k_i\ell_j}. \tag{3.13}$$

Applying (3.12) to the 2-tuples $(|k'|,k_n)$ and $(|\ell'|,\ell_n)$, we get

$$\frac{\big[|k'|+k_n+|\ell'|+\ell_n\big]_q!}{\big[|k'|+|\ell'|\big]_q!\,[k_n+\ell_n]_q!} \ge \frac{\big[|k'|+k_n\big]_q!}{\big[|k'|\big]_q!\,[k_n]_q!}\,\frac{\big[|\ell'|+\ell_n\big]_q!}{\big[|\ell'|\big]_q!\,[\ell_n]_q!}\, q^{|k'|\ell_n}. \tag{3.14}$$

Multiplying (3.13) by (3.14) yields (3.12). $\square$

Lemma 3.8. *Given $q\in\mathbb{C}^\times$, let*

$$u_q(k)=|q|^{\sum_{i<j} k_i k_j} \qquad (k=(k_1,\ldots,k_n)\in\mathbb{Z}^n_+).$$

For each $\rho > 0$, define a norm on $\mathscr{O}_q^{\mathrm{reg}}(\mathbb{C}^n)$ by

$$\|f\|_{\mathbb{B},\rho} = \sum_{k\in\mathbb{Z}_+^n} |c_k| \left(\frac{[k]_{|q|^2}!}{\big[|k|\big]_{|q|^2}!}\right)^{1/2} u_q(k)\rho^{|k|} \qquad \Big(f = \sum_{k\in\mathbb{Z}_+^n} c_k x^k \in \mathscr{O}_q^{\mathrm{reg}}(\mathbb{C}^n)\Big).$$

Then $\|\cdot\|_{\mathbb{B},\rho}$ is submultiplicative.

Proof. For each $k,\ell \in \mathbb{Z}_+^n$, let

$$c(k,\ell) = q^{-\sum_{i>j} k_i\ell_j}.$$

Then

$$x^k x^\ell = c(k,\ell) x^{k+\ell}$$

(see, e.g., [10,55]). By [106, Lemma 5.7], $\|\cdot\|_{\mathbb{B},\rho}$ is submultiplicative if and only if for each $k,\ell \in \mathbb{Z}_+^n$

$$\left(\frac{[k+\ell]_{|q|^2}!}{\big[|k+\ell|\big]_{|q|^2}!}\right)^{1/2} u_q(k+\ell)\,|c(k,\ell)| \le \left(\frac{[k]_{|q|^2}!\,[\ell]_{|q|^2}!}{\big[|k|\big]_{|q|^2}!\,\big[|\ell|\big]_{|q|^2}!}\right)^{1/2} u_q(k)u_q(\ell). \tag{3.15}$$

We have

$$\begin{aligned} u_q(k+\ell)\,|c(k,\ell)| &= |q|^{\sum_{i<j}(k_i+\ell_i)(k_j+\ell_j)} |q|^{-\sum_{i<j} k_j\ell_i} = |q|^{\sum_{i<j} k_ik_j+\ell_i\ell_j+k_i\ell_j};\\ u_q(k)u_q(\ell) &= |q|^{\sum_{i<j} k_ik_j+\ell_i\ell_j}. \end{aligned}$$

Therefore (3.15) is equivalent to

$$\left(\frac{[k+\ell]_{|q|^2}!}{\big[|k+\ell|\big]_{|q|^2}!}\right)^{1/2} \le \left(\frac{[k]_{|q|^2}!\,[\ell]_{|q|^2}!}{\big[|k|\big]_{|q|^2}!\,\big[|\ell|\big]_{|q|^2}!}\right)^{1/2} |q|^{-\sum_{i<j} k_i\ell_j}. \tag{3.16}$$

Raising (3.16) to the power -2 yields (3.12) with q replaced by $|q|^2$. The rest is clear. □

Theorem 3.9. *Let $q \in \mathbb{C}^\times$, and let $r \in (0,+\infty]$. The Fréchet space*

$$\mathscr{O}_q(\mathbb{B}_r^n) = \left\{ f = \sum_{k\in\mathbb{Z}_+^n} c_k x^k : \|f\|_{\mathbb{B},\rho} = \sum_{k\in\mathbb{Z}_+^n} |c_k| \left(\frac{[k]_{|q|^2}!}{\big[|k|\big]_{|q|^2}!}\right)^{1/2} u_q(k)\rho^{|k|} < \infty \ \forall \rho \in (0,r) \right\}$$

is a Fréchet-Arens-Michael algebra with respect to the multiplication uniquely determined by $x_ix_j = qx_jx_i$ ($i<j$). Moreover, the norms $\|\cdot\|_{\mathbb{B},\rho}$ are submultiplicative on $\mathscr{O}_q(\mathbb{B}_r^n)$.

Proof. Immediate from Lemma 3.8. □

Definition 3.10. The Fréchet algebra $\mathscr{O}_q(\mathbb{B}_r^n)$ will be called the *algebra of holomorphic functions on the quantum n-ball of radius r.*

In other words, $\mathscr{O}_q(\mathbb{B}_r^n)$ is the completion of $\mathscr{O}_q^{\mathrm{reg}}(\mathbb{C}^n)$ with respect to the family $\{\|\cdot\|_{\mathbb{B},\rho} : \rho \in (0,r)\}$ of submultiplicative norms. Letting $q=1$ in Definition 3.10 and comparing with (3.9), we see that the map $\mathscr{O}(\mathbb{B}_r^n) \to \mathscr{O}_1(\mathbb{B}_r^n)$ taking each $f \in \mathscr{O}(\mathbb{B}_r^n)$ to its Taylor series at 0 is a Fréchet algebra isomorphism.

Remark 3.11. At the moment, we still do not know whether $\mathscr{O}_q(\mathbb{B}_\infty^n) = \mathscr{O}_q(\mathbb{C}^n)$. This will be proved in Section 4.

Remark 3.12. One may wonder why we do not try to define q-deformed multiplications on the same Fréchet spaces $\mathscr{O}(\mathbb{D}_r^n)$ and $\mathscr{O}(\mathbb{B}_r^n)$. The answer is that such multiplications do not exist in general. To be more precise, if $|q|<1$ and $n\geq 2$, then there is no continuous multiplication $\star$ on $\mathscr{O}(\mathbb{D}_r^n)$ such that $x_i x_j = x_i \star x_j = q x_j \star x_i$ for all $i<j$ (where $x_1,\ldots,x_n$ are the coordinates on $\mathbb{C}^n$). Indeed, assume that such a multiplication exists. Then for each $\rho\in(0,r)$ there exist $C>0$ and $s\in(0,r)$ such that $\|f\star g\|_\rho \leq C\|f\|_s\|g\|_s$ ($f,g\in\mathscr{O}(\mathbb{D}_r^n)$). In particular, for each $m\in\mathbb{N}$ we have $\|x_2^m\star x_1^m\|_\rho \leq C\|x_2^m\|_s\|x_1^m\|_s = Cs^{2m}$. Since $\|x_2^m\star x_1^m\|_\rho = \|q^{-m^2}x_1^m x_2^m\|_\rho = |q|^{-m^2}\rho^{2m}$, we conclude that $|q|^{-m^2}\leq C(s/\rho)^{2m}$ for all m, which is a contradiction. A similar argument works for $\mathscr{O}(\mathbb{B}_r^n)$.

Our next goal is to extend the algebra isomorphism

$$\tau\colon \mathscr{O}_q^{\mathrm{reg}}(\mathbb{C}^n)\to\mathscr{O}_{q^{-1}}^{\mathrm{reg}}(\mathbb{C}^n),\quad x_i\mapsto x_{n-i}, \tag{3.17}$$

to $\mathscr{O}_q(\mathbb{D}_r^n)$ and $\mathscr{O}_q(\mathbb{B}_r^n)$. To this end, we need a lemma.

Lemma 3.13. *For each $q>0$ and each $k=(k_1,\ldots,k_n)\in\mathbb{Z}_+^n$, we have*

$$\frac{[k]_q!}{\big[|k|\big]_q!}\,q^{\sum_{i<j}k_ik_j} = \frac{[k]_{q^{-1}}!}{\big[|k|\big]_{q^{-1}}!}.$$

Proof. For each $m\in\mathbb{Z}_+$, we have $[m]_q = q^{m-1}[m]_{q^{-1}}$, and so

$$[m]_q! = q^{\frac{m(m-1)}{2}}[m]_{q^{-1}}!.$$

Therefore

$$\begin{aligned}\frac{[k]_q!}{\big[|k|\big]_q!}\,q^{\sum_{i<j}k_ik_j} &= \frac{q^{\sum_i\frac{k_i(k_i-1)}{2}}[k]_{q^{-1}}!}{q^{\frac{|k|(|k|-1)}{2}}\big[|k|\big]_{q^{-1}}!}\,q^{\sum_{i<j}k_ik_j}\\ &= \frac{q^{\frac{\sum_i k_i^2-\sum_i k_i}{2}}[k]_{q^{-1}}!}{q^{\frac{\sum_i k_i^2+2\sum_{i<j}k_ik_j-\sum_i k_i}{2}}\big[|k|\big]_{q^{-1}}!}\,q^{\sum_{i<j}k_ik_j} = \frac{[k]_{q^{-1}}!}{\big[|k|\big]_{q^{-1}}!}.\end{aligned}$$

□

Corollary 3.14. *For each $q\in\mathbb{C}^\times$, each $f\in\mathscr{O}_q(\mathbb{B}_r^n)$, and each $\rho\in(0,r)$, we have*

$$\|f\|_{\mathbb{B},\rho} = \sum_{k\in\mathbb{Z}_+^n}|c_k|\left(\frac{[k]_{|q|^{-2}}!}{\big[|k|\big]_{|q|^{-2}}!}\right)^{1/2}\rho^{|k|}.$$

Proof. Apply Lemma 3.13 with q replaced by $|q|^2$. □

Proposition 3.15. *For each $q\in\mathbb{C}^\times$ and each $r\in(0,+\infty]$, there exist topological algebra isomorphisms*

$$\begin{aligned}\tau_{\mathbb{D}}\colon \mathscr{O}_q(\mathbb{D}_r^n)\to\mathscr{O}_{q^{-1}}(\mathbb{D}_r^n),\quad x_i\mapsto x_{n-i};\\ \tau_{\mathbb{B}}\colon \mathscr{O}_q(\mathbb{B}_r^n)\to\mathscr{O}_{q^{-1}}(\mathbb{B}_r^n),\quad x_i\mapsto x_{n-i}.\end{aligned}$$

Moreover, for each $f\in\mathscr{O}_q(\mathbb{D}_r^n)$, each $g\in\mathscr{O}_q(\mathbb{B}_r^n)$, and each $\rho\in(0,r)$, we have

$$\|\tau_{\mathbb{D}}(f)\|_{\mathbb{D},\rho} = \|f\|_{\mathbb{D},\rho};\quad \|\tau_{\mathbb{B}}(g)\|_{\mathbb{B},\rho} = \|g\|_{\mathbb{B},\rho}. \tag{3.18}$$

Proof. For convenience, let us denote the norm $\|\cdot\|_{\mathbb{D},\rho}$ on $\mathscr{O}_q(\mathbb{D}_r^n)$ by $\|\cdot\|_{\mathbb{D},q,\rho}$. Similarly, we write $\|\cdot\|_{\mathbb{B},q,\rho}$ for the norm $\|\cdot\|_{\mathbb{B},\rho}$ on $\mathscr{O}_q(\mathbb{B}_r^n)$. In view of (3.17), it suffices to show that for each $f \in \mathscr{O}_q^{\mathrm{reg}}(\mathbb{C}^n)$ and each $\rho > 0$, we have

$$\|\tau(f)\|_{\mathbb{D},q^{-1},\rho} = \|f\|_{\mathbb{D},q,\rho}; \tag{3.19}$$

$$\|\tau(f)\|_{\mathbb{B},q^{-1},\rho} = \|f\|_{\mathbb{B},q,\rho}. \tag{3.20}$$

Without loss of generality, we may assume that $|q| \le 1$.

Observe that, for each $k = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n$, we have

$$\tau(x^k) = x_n^{k_1} \cdots x_1^{k_n} = q^{\sum_{i<j} k_i k_j} x_1^{k_n} \cdots x_n^{k_1}.$$

Therefore, for each $f = \sum_k c_k x^k \in \mathscr{O}_q^{\mathrm{reg}}(\mathbb{C}^n)$,

$$\|\tau(f)\|_{\mathbb{D},q^{-1},\rho} = \sum_k |c_k| w_q(k) \rho^{|k|} = \|f\|_{\mathbb{D},q,\rho}.$$

This proves (3.19). Similarly, using Corollary 3.14, we obtain

$$\|\tau(f)\|_{\mathbb{B},q^{-1},\rho} = \sum_k |c_k| u_q(k) \left(\frac{[k]_{|q|^2}!}{\big[|k|\big]_{|q|^2}!} \right)^{1/2} \rho^{|k|} = \|f\|_{\mathbb{B},q,\rho}.$$

This proves (3.20) and completes the proof. □

4. A q-analog of Poincaré's theorem

A classical result of H. Poincaré [111] (see also [134, Theorem 2.7]) asserts that the polydisk $\mathbb{D}_r^n$ and the ball $\mathbb{B}_r^n$ are not biholomorphically equivalent (unless $n = 1$ or $r = \infty$). By O. Forster's theorem [51] (see also [57, V.7]), the category of Stein spaces is anti-equivalent to the category of Stein algebras (i.e., Fréchet algebras of the form $\mathscr{O}(X)$, where X is a Stein space) via the functor $X \mapsto \mathscr{O}(X)$. Therefore, when translated into the dual language, Poincaré's theorem states that the Fréchet algebras $\mathscr{O}(\mathbb{D}_r^n)$ and $\mathscr{O}(\mathbb{B}_r^n)$ are not topologically isomorphic.

A natural question is whether or not Poincaré's theorem has a q-analog, i.e., whether or not the Fréchet algebras $\mathscr{O}_q(\mathbb{D}_r^n)$ and $\mathscr{O}_q(\mathbb{B}_r^n)$ are topologically isomorphic. The goal of this section is to answer the above question.

Lemma 4.1. *For each $q \in (0,1)$ and each $k \in \mathbb{Z}_+^n$, we have*

$$(q;q)_\infty^n \le \frac{[k]_q!}{\big[|k|\big]_q!} \le 1.$$

Proof. Let $k = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n$. Then

$$\frac{[k]_q!}{\big[|k|\big]_q!} = \frac{\prod_{i=1}^n [k_i]_q!}{[k_1 + \cdots + k_n]_q!} = \frac{\prod_{i=1}^n \prod_{j=1}^{k_i} (1 - q^j)}{\prod_{p=1}^{k_1+\cdots+k_n} (1 - q^p)} = \prod_{i=1}^n \prod_{j=1}^{k_i} \frac{1 - q^j}{1 - q^{k_1+\cdots+k_{i-1}+j}} \le 1.$$

On the other hand,

$$\frac{[k]_q!}{\big[|k|\big]_q!} \ge \prod_{i=1}^n \prod_{j=1}^{k_i} (1 - q^j) \ge \prod_{i=1}^n \prod_{j=1}^{\infty} (1 - q^j) = (q;q)_\infty^n.$$ □

Theorem 4.2. *Let $q \in \mathbb{C}^\times$, $|q| \neq 1$, and let $r \in (0,+\infty]$. Then $\mathscr{O}_q(\mathbb{D}^n_r) = \mathscr{O}_q(\mathbb{B}^n_r)$ as vector subspaces of $\mathbb{C}[[x_1,\ldots,x_n]]$ and as Fréchet algebras. Moreover, for each $\rho \in (0,r)$ we have*

$$(|q|^{-2};|q|^{-2})_\infty^{n/2}\|\cdot\|_{\mathbb{D},\rho} \le \|\cdot\|_{\mathbb{B},\rho} \le \|\cdot\|_{\mathbb{D},\rho} \qquad (|q|>1); \tag{4.1}$$

$$(|q|^{2};|q|^{2})_\infty^{n/2}\|\cdot\|_{\mathbb{D},\rho} \le \|\cdot\|_{\mathbb{B},\rho} \le \|\cdot\|_{\mathbb{D},\rho} \qquad (|q|<1). \tag{4.2}$$

Proof. To prove the result, it suffices to show that (4.1) and (4.2) hold on $\mathscr{O}_q^{\mathrm{reg}}(\mathbb{C}^n)$. Suppose first that $|q|>1$. Applying Corollary 3.14 and Lemma 4.1, for each $f=\sum_k c_k x^k \in \mathscr{O}_q^{\mathrm{reg}}(\mathbb{C}^n)$ we obtain

$$\|f\|_{\mathbb{B},\rho} = \sum_k |c_k| \left(\frac{[k]_{|q|^{-2}}!}{\big[|k|\big]_{|q|^{-2}}!}\right)^{1/2} \rho^{|k|} \le \sum_k |c_k|\rho^{|k|} = \|f\|_{\mathbb{D},\rho};$$

$$\begin{aligned}\|f\|_{\mathbb{B},\rho} &= \sum_k |c_k| \left(\frac{[k]_{|q|^{-2}}!}{\big[|k|\big]_{|q|^{-2}}!}\right)^{1/2} \rho^{|k|}\\ &\ge (|q|^{-2};|q|^{-2})_\infty^{n/2} \sum_k |c_k|\rho^{|k|} = (|q|^{-2};|q|^{-2})_\infty^{n/2}\|f\|_{\mathbb{D},\rho}.\end{aligned}$$

This completes the proof in the case where $|q|>1$. The case $|q|<1$ is reduced to the previous one by applying (3.18). ☐

Corollary 4.3. *For each $q \in \mathbb{C}^\times$, we have $\mathscr{O}_q(\mathbb{B}^n_\infty) = \mathscr{O}_q(\mathbb{C}^n)$ as vector subspaces of $\mathbb{C}[[x_1,\ldots,x]]$ and as Fréchet algebras.*

Proof. If $|q|\neq 1$, then the result follows from Theorem 4.2. Suppose now that $|q|=1$. By looking at (2.2) and (3.1), we see that the map from $\mathscr{O}(\mathbb{C}^n)$ to $\mathscr{O}_q(\mathbb{C}^n)$ that takes each $f \in \mathscr{O}(\mathbb{C}^n)$ to its Taylor series at 0 is a Fréchet space isomorphism. On the other hand, Proposition 3.5 yields a similar topological isomorphism between the underlying Fréchet spaces of $\mathscr{O}(\mathbb{C}^n)$ and $\mathscr{O}_q(\mathbb{B}^n_\infty)$. By composing these isomorphisms, we obtain a Fréchet space isomorphism between $\mathscr{O}_q(\mathbb{C}^n)$ and $\mathscr{O}_q(\mathbb{B}^n_\infty)$ taking x^k to x^k ($k \in \mathbb{Z}^n_+$). Clearly, this is an algebra isomorphism. ☐

Remark 4.4. We have already mentioned that, for each $q \in \mathbb{C}^\times$, the monomials x^k ($k \in \mathbb{Z}^n_+$) form a basis of $\mathscr{O}_q^{\mathrm{reg}}(\mathbb{C}^n)$. Hence we have a vector space isomorphism $\mathscr{O}^{\mathrm{reg}}(\mathbb{C}^n) \to \mathscr{O}_q^{\mathrm{reg}}(\mathbb{C}^n)$ given by $x^k \mapsto x^k$. It is natural to ask whether a similar result holds for $\mathscr{O}_q(\mathbb{D}^n_r)$ and $\mathscr{O}_q(\mathbb{B}^n_r)$, i.e., whether there exist Fréchet space isomorphisms

$$\varphi_1\colon \mathscr{O}(\mathbb{D}^n_r) \to \mathscr{O}_q(\mathbb{D}^n_r), \quad \varphi_2\colon \mathscr{O}(\mathbb{B}^n_r) \to \mathscr{O}_q(\mathbb{B}^n_r),$$

that take each holomorphic function f to its Taylor series at 0. If $|q|=1$, then, comparing (3.1) with Definition 3.1 and (3.9) with Definition 3.10, respectively, we see that both φ_1 and φ_2 are Fréchet space isomorphisms. The same argument shows that φ_1 is a Fréchet space isomorphism whenever $|q|\ge 1$. If $|q|<1$, then φ_1 is a continuous linear map (because $w_q(k)\le 1$), but is not an isomorphism by Remark 3.12. For the same reason, if $|q|<1$, then φ_2 is a continuous linear map, but is not an isomorphism. Finally, if $|q|>1$, then φ_2 is not well defined (unless $r=\infty$). Indeed, if φ_2 existed, then we would have a chain of linear maps

$$\mathscr{O}(\mathbb{B}^n_r) \xrightarrow{\varphi_2} \mathscr{O}_q(\mathbb{B}^n_r) = \mathscr{O}_q(\mathbb{D}^n_r) \xrightarrow{\varphi_1^{-1}} \mathscr{O}(\mathbb{D}^n_r),$$

and the composite map $\varphi\colon \mathscr{O}(\mathbb{B}_r^n) \to \mathscr{O}(\mathbb{D}_r^n)$ would take each z^k to itself. Therefore φ would be an inverse for the restriction map $\mathscr{O}(\mathbb{D}_r^n) \to \mathscr{O}(\mathbb{B}_r^n)$, which is a contradiction.

Now let us turn to a more difficult and more interesting case $|q| = 1$. To prove our q-version of Poincaré's theorem, we will need to extend the notion of joint spectral radius in Banach algebras (see [96, V.35 and Comments to Chapter V]) to the setting of Arens-Michael algebras.

Definition 4.5. Let A be an Arens-Michael algebra, and let $\{\|\cdot\|_\lambda : \lambda \in \Lambda\}$ be a directed defining family of submultiplicative seminorms on A. Given an n-tuple $a = (a_1, \ldots, a_n) \in A^n$, we define the *joint ℓ^p-spectral radius $r_p^A(a)$* by

$$\begin{aligned} r_p^A(a) &= \sup_{\lambda\in\Lambda} \lim_{d\to\infty} \Big(\sum_{\alpha\in W_{n,d}} \|a_\alpha\|_\lambda^p \Big)^{1/pd} \quad \text{for } 1 \le p < \infty; \\ r_\infty^A(a) &= \sup_{\lambda\in\Lambda} \lim_{d\to\infty} \Big(\sup_{\alpha\in W_{n,d}} \|a_\alpha\|_\lambda \Big)^{1/d}. \end{aligned} \tag{4.3}$$

Remark 4.6. The joint ℓ^∞-spectral radius was studied by A. Sołtysiak [156] in the case where the a_j's commute, but A is not necessarily locally m-convex.

By [96, C.35.2], the limits in (4.3) always exist. In contrast to the Banach algebra case, it may happen that $r_p^A(a) = +\infty$. For example, if $A = \mathscr{O}(\mathbb{C})$ and $z \in A$ is the complex coordinate, then an easy computation shows that $r_p^A(z) = +\infty$ for all p (see also Examples 4.11 and 4.12 below).

Proposition 4.7. *The definition of $r_p^A(a)$ does not depend on the choice of a directed defining family of submultiplicative seminorms on A.*

Proof. Let $S = \{\|\cdot\|_\lambda : \lambda \in \Lambda\}$ and $S' = \{\|\cdot\|_\mu : \mu \in \Lambda'\}$ be two directed defining families of submultiplicative seminorms on A, and let $r_p^A(a; S)$ and $r_p^A(a; S')$ denote the respective joint spectral radii. Then for each $\mu \in \Lambda'$ there exist $\lambda \in \Lambda$ and $C > 0$ such that $\|\cdot\|_\mu \le C\|\cdot\|_\lambda$ on A. Therefore for each $p \in [1, +\infty)$ we obtain

$$\lim_{d\to\infty} \Big(\sum_{\alpha\in W_{n,d}} \|a_\alpha\|_\mu^p \Big)^{1/pd} \le \lim_{d\to\infty} C^{1/d} \Big(\sum_{\alpha\in W_{n,d}} \|a_\alpha\|_\lambda^p \Big)^{1/pd} \le r_p^A(a; S),$$

whence $r_p^A(a; S') \le r_p^A(a; S)$. For $p = \infty$, the computation is similar. Since S and S' are equivalent, the result follows. ☐

Given an algebra A, an n-tuple $a = (a_1, \ldots, a_n) \in A^n$, and an algebra homomorphism $\varphi\colon A \to B$, we denote by $\varphi(a)$ the n-tuple $(\varphi(a_1), \ldots, \varphi(a_n)) \in B^n$.

Proposition 4.8. *Let A and B be Arens-Michael algebras, and let $\varphi\colon A \to B$ be a continuous homomorphism. Then for each $a \in A^n$ we have $r_p^B(\varphi(a)) \le r_p^A(a)$.*

Proof. Let $\{\|\cdot\|_\lambda : \lambda \in \Lambda\}$ and $\{\|\cdot\|_\mu : \mu \in \Lambda'\}$ be directed defining families of submultiplicative seminorms on A and B, respectively. Then for each $\mu \in \Lambda'$ there exist $\lambda \in \Lambda$ and $C > 0$ such that for each $a \in A$ we have $\|\varphi(a)\|_\mu \le C\|a\|_\lambda$. Let now $a \in A^n$ and

$p \in [1, +\infty)$. We have

$$\lim_{d\to\infty}\Big(\sum_{\alpha\in W_{n,d}} \|\varphi(a)_\alpha\|_\mu^p\Big)^{1/pd} = \lim_{d\to\infty}\Big(\sum_{\alpha\in W_{n,d}} \|\varphi(a_\alpha)\|_\mu^p\Big)^{1/pd}$$
$$\le \lim_{d\to\infty} C^{1/d}\Big(\sum_{\alpha\in W_{n,d}} \|a_\alpha\|_\lambda^p\Big)^{1/pd} \le r_p^A(a).$$

For $p = \infty$, the computation is similar. The rest is clear. □

Corollary 4.9. *Let A and B be Arens-Michael algebras, and let $\varphi \colon A \to B$ be a topological algebra isomorphism. Then for each $a \in A^n$ we have $r_p^B(\varphi(a)) = r_p^A(a)$.*

Remark 4.10. If A is a commutative Banach algebra, then for each $a \in A^n$ we have

$$r_p^A(a) = \sup\{\|z\|_p : z \in \sigma_A(a)\}, \tag{4.4}$$

where $\sigma_A(a)$ is the joint spectrum of a and $\|\cdot\|_p$ is the ℓ^p-norm on $\mathbb{C}^n$ (see [155] or [96, Theorems 35.5 and 35.6]). This result easily extends to commutative Arens-Michael algebras. Indeed, let A be a commutative Arens-Michael algebra, and let $\{\|\cdot\|_\lambda : \lambda \in \Lambda\}$ be a directed defining family of submultiplicative seminorms on A. For each $\lambda \in \Lambda$, let A_λ denote the completion of A with respect to $\|\cdot\|_\lambda$. By definition, we have $r_p^A(a) = \sup_\lambda r_p^{A_\lambda}(a_\lambda)$, where a_λ is the canonical image of a in A_λ^n. On the other hand, a standard argument (cf. [60, Proposition 5.1.8]) shows that $\sigma_A(a) = \bigcup_\lambda \sigma_{A_\lambda}(a_\lambda)$. Applying now (4.4) to each a_λ and taking then the supremum over λ, we get the result. In the case where $p = \infty$, a more general fact was proved by A. Sołtysiak [156].

Let us introduce some notation. Given $\alpha \in W_n$ and $i \in \{1, \ldots, n\}$, let

$$\mathrm{p}_i(\alpha) = |\alpha^{-1}(i)|.$$

Thus we have a map

$$\mathrm{p} \colon W_n \to \mathbb{Z}_+^n, \quad \mathrm{p}(\alpha) = (\mathrm{p}_1(\alpha), \ldots, \mathrm{p}_n(\alpha)). \tag{4.5}$$

Observe that, for each $k \in \mathbb{Z}_+^n$, we have

$$|\mathrm{p}^{-1}(k)| = \frac{|k|!}{k!}. \tag{4.6}$$

Let now $q \in \mathbb{C}^\times$, and let $x_1, \ldots, x_n$ be the canonical generators of $\mathscr{O}_q^{\mathrm{reg}}(\mathbb{C}^n)$. Then for each $\alpha \in W_n$ there exists a unique $\mathrm{t}(\alpha) \in \mathbb{C}^\times$ such that

$$x_\alpha = \mathrm{t}(\alpha) x^{\mathrm{p}(\alpha)}. \tag{4.7}$$

An explicit formula for $\mathrm{t}(\alpha)$ will be given in Lemma 7.8; at the moment, let us only observe that $\mathrm{t}(\alpha)$ is an integer power of q.

The following two examples will be crucial for what follows.

Example 4.11. Let $|q| = 1$, and let $x = (x_1, \ldots, x_n) \in \mathscr{O}_q(\mathbb{D}_r^n)^n$. We claim that

$$r_2^{\mathscr{O}_q(\mathbb{D}_r^n)}(x) = r\sqrt{n}. \tag{4.8}$$

Indeed, for each $\rho \in (0,r)$ we have

$$\lim_{d\to\infty}\Big(\sum_{\alpha\in W_{n,d}} \|x_\alpha\|^2_{\mathbb{D},\rho}\Big)^{1/2d} = \lim_{d\to\infty}\Big(\sum_{\alpha\in W_{n,d}} \|x^{\mathrm{p}(\alpha)}\|^2_{\mathbb{D},\rho}\Big)^{1/2d} = \lim_{d\to\infty}\Big(\sum_{\alpha\in W_{n,d}} \rho^{2d}\Big)^{1/2d}$$
$$= \rho \lim_{d\to\infty} |W_{n,d}|^{1/2d} = \rho \lim_{d\to\infty} (n^d)^{1/2d} = \rho\sqrt{n}.$$

Taking the supremum over ρ yields (4.8).

Example 4.12. Let $|q|=1$, and let $x=(x_1,\ldots,x_n)\in \mathscr{O}_q(\mathbb{B}^n_r)^n$. We claim that

$$r_2^{\mathscr{O}_q(\mathbb{B}^n_r)}(x) = r. \tag{4.9}$$

To see this, observe that for each $d\in\mathbb{Z}_+$ we have

$$\big|(\mathbb{Z}^n_+)_d\big| = \binom{d+n-1}{n-1} \le (d+1)(d+2)\cdots(d+n-1) \le (d+n-1)^{n-1}. \tag{4.10}$$

Hence for each $\rho\in(0,r)$ we obtain

$$\lim_{d\to\infty}\Big(\sum_{\alpha\in W_{n,d}} \|x_\alpha\|^2_{\mathbb{B},\rho}\Big)^{1/2d} = \lim_{d\to\infty}\Big(\sum_{\alpha\in W_{n,d}} \|x^{\mathrm{p}(\alpha)}\|^2_{\mathbb{B},\rho}\Big)^{1/2d}$$
$$= \lim_{d\to\infty}\Big(\sum_{k\in(\mathbb{Z}^n_+)_d} \frac{|k|!}{k!}\|x^k\|^2_{\mathbb{B},\rho}\Big)^{1/2d}$$
$$= \lim_{d\to\infty}\Big(\sum_{k\in(\mathbb{Z}^n_+)_d} \rho^{2d}\Big)^{1/2d} = \rho \lim_{d\to\infty} |(\mathbb{Z}^n_+)_d|^{1/2d} = \rho.$$

Taking the supremum over ρ yields (4.9).

Remark 4.13. We have already noticed (see Remark 4.4) that, if $|q|=1$, then $\mathscr{O}_q(\mathbb{D}^n_r) = \mathscr{O}(\mathbb{D}^n_r)$ and $\mathscr{O}_q(\mathbb{B}^n_r) = \mathscr{O}(\mathbb{B}^n_r)$ as locally convex spaces. Also, observe that $\|x_\alpha\|_{\mathbb{D},\rho}$ and $\|x_\alpha\|_{\mathbb{B},\rho}$ do not depend on q provided that $|q|=1$. Hence $r_2^{\mathscr{O}_q(\mathbb{D}^n_r)}(x)$ and $r_2^{\mathscr{O}_q(\mathbb{B}^n_r)}(x)$ do not depend on q. On the other hand, it is easy to show that $\sigma_{\mathscr{O}(\mathbb{D}^n_r)}(x) = \mathbb{D}^n_r$ and $\sigma_{\mathscr{O}(\mathbb{B}^n_r)}(x) = \mathbb{B}^n_r$. Applying now (4.4), we obtain

$$r_2^{\mathscr{O}_q(\mathbb{D}^n_r)}(x) = \sup\{\|z\|_2 : z\in\mathbb{D}^n_r\} = r\sqrt{n};$$
$$r_2^{\mathscr{O}_q(\mathbb{B}^n_r)}(x) = \sup\{\|z\|_2 : z\in\mathbb{B}^n_r\} = r,$$

which yields an alternative proof of (4.8) and (4.9).

Although the algebras $\mathscr{O}_q(\mathbb{D}^n_r)$ and $\mathscr{O}_q(\mathbb{B}^n_r)$ are not graded in the purely algebraic sense, it will be convenient to introduce the following terminology. Let A denote either $\mathscr{O}_q(\mathbb{D}^n_r)$ or $\mathscr{O}_q(\mathbb{B}^n_r)$, and let

$$A_i = \operatorname{span}\{x^k : |k| = i\} \qquad (i\in\mathbb{Z}_+).$$

Then each $a\in A$ can be uniquely written as $a=\sum_{i=0}^\infty a_i$, where $a_i\in A_i$ and the series absolutely converges in A. The element a_i will be called the *ith homogeneous component* of a. Explicitly, if $a=\sum_k c_k x^k$, then $a_i=\sum_{|k|=i} c_k x^k$. We clearly have $A_iA_j\subseteq A_{i+j}$ for all $i,j\in\mathbb{Z}_+$. Observe also that for each $a,b\in A$ and each $\ell\in\mathbb{Z}_+$ we have

$$(ab)_\ell = \sum_{i+j=\ell} a_i b_j.$$

Let also

$$A_{\geq i} = \overline{\bigoplus_{j\geq i} A_j} = \overline{\operatorname{span}}\{x^k : |k| \geq i\} \qquad (i \in \mathbb{Z}_+).$$

Obviously,

$$A_{\geq i}A_{\geq j} \subseteq A_{\geq (i+j)} \qquad (i, j \in \mathbb{Z}_+). \tag{4.11}$$

Here is the main result of this section.

Theorem 4.14. *If $|q| = 1$, $n \geq 2$, and $r < \infty$, then the Fréchet algebras $\mathscr{O}_q(\mathbb{D}^n_r)$ and $\mathscr{O}_q(\mathbb{B}^n_r)$ are not topologically isomorphic.*

Proof. If $q = 1$, then the result follows from the classical Poincaré theorem (see the beginning of this section). Thus we may suppose that $q \neq 1$. Let $A = \mathscr{O}_q(\mathbb{D}^n_r)$, $B = \mathscr{O}_q(\mathbb{B}^n_r)$, and assume, towards a contradiction, that $\varphi\colon B \to A$ is a topological algebra isomorphism. For each $i = 1, \dots, n$, let $f_i = \varphi(x_i) \in A$ and $g_i = \varphi^{-1}(x_i) \in B$. Given $k \in \mathbb{Z}_+$, let $f_{i,k}$ (respectively, $g_{i,k}$) denote the kth homogeneous component of f_i (respectively, g_i). We claim that

$$f_{i,0} = g_{i,0} = 0 \qquad (i = 1, \dots, n). \tag{4.12}$$

Indeed, assume that $f_{i,0} \neq 0$ for some i, and fix any $j \neq i$. Then we have

$$f_i f_j = q' f_j f_i \qquad (\text{where } q' = q \text{ or } q' = q^{-1}). \tag{4.13}$$

Let now $k = \min\{\ell \in \mathbb{Z}_+ : f_{j,\ell} \neq 0\}$. Taking the kth homogeneous components of (4.13), we obtain $f_{i,0}f_{j,k} = q' f_{j,k} f_{i,0}$, whence $f_{j,k} = 0$. The resulting contradiction implies that $f_{i,0} = 0$ for all i. A similar argument shows that $g_{i,0} = 0$ for all i.

Thus for each $i = 1, \dots, n$ we have $\varphi(x_i) \in A_{\geq 1}$. Using (4.11), we see that for each $k \in \mathbb{Z}^n_+$

$$\varphi(x^k) = \varphi(x_1)^{k_1} \cdots \varphi(x_n)^{k_n} \in A^{|k|}_{\geq 1} \subseteq A_{\geq |k|}.$$

Similarly, $\varphi^{-1}(x^k) \in B_{\geq |k|}$. Hence for each $m \in \mathbb{Z}_+$ we have

$$\varphi(B_{\geq m}) \subseteq A_{\geq m}, \quad \varphi^{-1}(A_{\geq m}) \subseteq B_{\geq m}. \tag{4.14}$$

Let now $\mathbb{M}_n$ denote the algebra of all complex $n \times n$-matrices, and let $\alpha = (\alpha_{ij}) \in \mathbb{M}_n$ and $\beta = (\beta_{ij}) \in \mathbb{M}_n$ be such that

$$f_{i,1} = \sum_j \alpha_{ij} x_j, \quad g_{i,1} = \sum_j \beta_{ij} x_j. \tag{4.15}$$

Using (4.12) and (4.14), we obtain

$$\begin{aligned} x_i = \varphi(g_i) \in \varphi(g_{i,1} + B_{\geq 2}) \subseteq \varphi(g_{i,1}) + A_{\geq 2} &= \sum_j \beta_{ij} f_j + A_{\geq 2} \\ = \sum_j \beta_{ij} f_{j,1} + A_{\geq 2} = \sum_{j,k} \beta_{ij}\alpha_{jk} x_k + A_{\geq 2} &= \sum_k (\beta\alpha)_{ik} x_k + A_{\geq 2}. \end{aligned}$$

Hence $\beta\alpha = 1$ in $\mathbb{M}_n$. In particular, α is invertible.

Fix now $i, j \in \{1, \dots, n\}$ with $i < j$. Since $f_i f_j = q f_j f_i$ and $f_{i,0} = f_{j,0} = 0$, it follows that $f_{i,1}f_{j,1} = q f_{j,1} f_{i,1}$. Equivalently,

$$\sum_k \alpha_{ik} x_k \sum_\ell \alpha_{j\ell} x_\ell = q \sum_k \alpha_{jk} x_k \sum_\ell \alpha_{i\ell} x_\ell.$$

Comparing the coefficients at x_m^2 yields $\alpha_{im}\alpha_{jm} = q\alpha_{jm}\alpha_{im}$, whence $\alpha_{im}\alpha_{jm} = 0$ for all $m = 1, \dots, n$ and for all $i < j$. In other words, each column of α contains at most one nonzero element. Since α is invertible, we conclude that there exists a permutation σ of $\{1, \dots, n\}$ such that $\alpha_{ij} = 0$ whenever $i \neq \sigma(j)$, and $\alpha_{\sigma(j)j} \neq 0$. Let $\tau = \sigma^{-1}$, and let $\lambda_i = \alpha_{i\tau(i)}$. Since $\beta = \alpha^{-1}$, it follows that

$$\alpha_{i\tau(i)} = \lambda_i \neq 0, \quad \alpha_{ij} = 0 \quad (j \neq \tau(i));$$
$$\beta_{i\sigma(i)} = \lambda_{\sigma(i)}^{-1} \neq 0, \quad \beta_{ij} = 0 \quad (j \neq \sigma(i)).$$

Together with (4.12) and (4.15), this implies that

$$\varphi(x_i) = f_i \in \lambda_i x_{\tau(i)} + A_{\geq 2}, \quad \varphi^{-1}(x_i) = g_i \in \lambda_{\sigma(i)}^{-1} x_{\sigma(i)} + B_{\geq 2}. \tag{4.16}$$

Therefore for each $d \in \mathbb{Z}_+$ we have

$$\varphi(x_i^d) \in \lambda_i^d x_{\tau(i)}^d + A_{\geq (d+1)}, \quad \varphi^{-1}(x_i^d) \in \lambda_{\sigma(i)}^{-d} x_{\sigma(i)}^d + B_{\geq (d+1)},$$

whence for each $\rho \in (0, r)$ we obtain

$$\|\varphi(x_i^d)\|_{\mathbb{D},\rho} \geq \|\lambda_i^d x_{\tau(i)}^d\|_{\mathbb{D},\rho} = |\lambda_i|^d \rho^d, \tag{4.17}$$
$$\|\varphi^{-1}(x_i^d)\|_{\mathbb{B},\rho} \geq \|\lambda_{\sigma(i)}^{-d} x_{\sigma(i)}^d\|_{\mathbb{B},\rho} = |\lambda_{\sigma(i)}^{-1}|^d \rho^d. \tag{4.18}$$

Fix now $\rho \in (0, r)$, and choose $\omega(\rho) \in (\rho, r)$ and $C(\rho) > 0$ such that

$$\|\varphi(b)\|_{\mathbb{D},\rho} \leq C(\rho) \|b\|_{\mathbb{B},\omega(\rho)} \qquad (b \in B).$$

Letting $b = x_i^d$ and using (4.17), we see that

$$|\lambda_i|^d \rho^d \leq \|\varphi(x_i^d)\|_{\mathbb{D},\rho} \leq C(\rho) \|x_i^d\|_{\mathbb{B},\omega(\rho)} = C(\rho)\omega(\rho)^d. \tag{4.19}$$

Raising (4.19) to the power $1/d$ and letting then $d \to \infty$, we conclude that $|\lambda_i| \leq \omega(\rho)/\rho$. Letting now $\rho \to r$, we obtain $|\lambda_i| \leq 1$. Applying the same argument to φ^{-1} and using (4.18) instead of (4.17), we see that $|\lambda_{\sigma(i)}^{-1}| \leq 1$. Finally, $|\lambda_i| = 1$ for all $i = 1, \dots, n$.

Given $\alpha = (\alpha_1, \dots, \alpha_d) \in W_n$, let $\tau(\alpha) = (\tau(\alpha_1), \dots, \tau(\alpha_d)) \in W_n$. Using again (4.16), we see that

$$f_\alpha \in \lambda_\alpha x_{\tau(\alpha)} + A_{\geq (|\alpha|+1)} \qquad (\alpha \in W_n),$$

whence

$$\|f_\alpha\|_{\mathbb{D},\rho} \geq \|\lambda_\alpha x_{\tau(\alpha)}\|_{\mathbb{D},\rho} = \|x_\alpha\|_{\mathbb{D},\rho} \qquad (\alpha \in W_n, \ \rho \in (0, r)).$$

Therefore $r_2^A(f_1, \dots, f_n) \geq r_2^A(x_1, \dots, x_n)$. Combining this with (4.8), (4.9), and using Corollary 4.9, we see that

$$r = r_2^B(x_1, \dots, x_n) = r_2^A(f_1, \dots, f_n) \geq r_2^A(x_1, \dots, x_n) = r\sqrt{n}.$$

The resulting contradiction completes the proof. □

We conclude this section with an open problem related to the notion of an HFG algebra [109, 110]. Let $\mathscr{F}_n$ denote the Arens-Michael envelope (2.4) of the free algebra F_n. A Fréchet algebra A is said to be *holomorphically finitely generated* (HFG for short) if A is isomorphic to a quotient of $\mathscr{F}_n$ for some n. There is also an "internal" definition given in terms of J. L. Taylor's free functional calculus. By [110, Theorem 3.22], a commutative Fréchet-Arens-Michael algebra is holomorphically finitely generated if and only if it is topologically isomorphic to $\mathscr{O}(X)$ for some Stein space $(X, \mathscr{O}_X)$ of finite embedding dimension. Together with O. Forster's theorem [51], this implies that the category of commutative HFG algebras is anti-equivalent to the category of Stein spaces

of finite embedding dimension. There are many natural examples of noncommutative HFG algebras [110, Section 7]. For instance, $\mathscr{O}_q(\mathbb{D}^n_r)$ and $\mathscr{F}(\mathbb{D}^n_r)$ are HFG algebras. By Theorem 4.2, $\mathscr{O}_q(\mathbb{B}^n_r)$ is an HFG algebra provided that $|q| \neq 1$.

Problem 4.15. *Is $\mathscr{O}_q(\mathbb{B}^n_r)$ an HFG algebra in the case where $|q| = 1,\ q \neq 1$?*

5. Quantum ball à la Vaksman

In this section we establish a relation between $\mathscr{O}_q(\mathbb{B}^n_r)$ and L. L. Vaksman's algebra $C_q(\bar{\mathbb{B}}^n)$, which is a natural q-analog of the algebra $C(\bar{\mathbb{B}}^n)$ of continuous functions on the closed unit ball $\bar{\mathbb{B}}^n = \bar{\mathbb{B}}^n_1$ [167]. To motivate the construction, let us start with the classical situation. Let $\operatorname{Fun}(\mathbb{C}^n)$ denote the algebra of all $\mathbb{C}$-valued functions on $\mathbb{C}^n$. There is a natural involution on $\operatorname{Fun}(\mathbb{C}^n)$ given by $f^*(z) = \overline{f(z)}$. Let $\operatorname{Pol}(\mathbb{C}^n)$ denote the $*$-subalgebra of $\operatorname{Fun}(\mathbb{C}^n)$ generated by the coordinates $z_1, \ldots, z_n$ on $\mathbb{C}^n$. Clearly, we have an algebra isomorphism $\operatorname{Pol}(\mathbb{C}^n) \cong \mathbb{C}[z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n]$. By the Stone-Weierstrass theorem, the completion of $\operatorname{Pol}(\mathbb{C}^n)$ with respect to the uniform norm $\|f\|^\infty_{\mathbb{B}} = \sup_{z \in \bar{\mathbb{B}}^n} |f(z)|$ is $C(\bar{\mathbb{B}}^n)$. For each $\rho > 0$ and each $f \in \mathbb{C}[z_1, \ldots, z_n]$, let

$$\|f\|^\infty_{\mathbb{B},\rho} = \sup_{z \in \bar{\mathbb{B}}^n_\rho} |f(z)| = \|\gamma_\rho(f)\|^\infty_{\mathbb{B}},$$

where γ_ρ is the automorphism of $\mathbb{C}[z_1, \ldots, z_n]$ uniquely determined by $\gamma_\rho(z_i) = \rho z_i$ $(i = 1, \ldots, n)$. Clearly, the completion of $\mathbb{C}[z_1, \ldots, z_n]$ with respect to the family $\{\| \cdot \|_{\mathbb{B},\rho} : \rho \in (0, r)\}$ of norms is topologically isomorphic to $\mathscr{O}(\mathbb{B}^n_r)$.

Now let us "quantize" the above data. Fix $q \in (0, 1)$, and denote by $\operatorname{Pol}_q(\mathbb{C}^n)$ the $*$-algebra generated (as a $*$-algebra) by n elements $x_1, \ldots, x_n$ subject to the relations

$$\begin{aligned} &x_i x_j = q x_j x_i \quad (i < j);\\ &x_i^* x_j = q x_j x_i^* \quad (i \neq j);\\ &x_i^* x_i = q^2 x_i x_i^* + (1 - q^2)\Big(1 - \sum_{k>i} x_k x_k^*\Big). \end{aligned} \tag{5.1}$$

Clearly, for $q = 1$ we have $\operatorname{Pol}_q(\mathbb{C}^n) \cong \operatorname{Pol}(\mathbb{C}^n)$. The algebra $\operatorname{Pol}_q(\mathbb{C}^n)$ was introduced by W. Pusz and S. L. Woronowicz [133], although they used different $*$-generators $a_1, \ldots, a_n$ given by $a_i = (1 - q^2)^{-1/2} x_i^*$. Relations (5.1) divided by $1 - q^2$ and written in terms of the a_i's are called the "twisted canonical commutation relations", and the algebra $A_q = \operatorname{Pol}_q(\mathbb{C}^n)$ defined in terms of the a_i's is sometimes called the "quantum Weyl algebra" (see, e.g., [4, 66, 79, 176]). Note that, while $\operatorname{Pol}_q(\mathbb{C}^n)$ becomes $\operatorname{Pol}(\mathbb{C}^n)$ for $q = 1$, A_q becomes the Weyl algebra. The idea to use the generators x_i instead of the a_i's and to consider $\operatorname{Pol}_q(\mathbb{C}^n)$ as a q-analog of $\operatorname{Pol}(\mathbb{C}^n)$ is probably due to Vaksman [163]; the one-dimensional case was considered in [76]. The algebra $\operatorname{Pol}_q(\mathbb{C}^n)$ serves as a basic example in the general theory of quantum bounded symmetric domains developed by Vaksman and his collaborators (see [166, 168] and references therein).

Let H be a Hilbert space with an orthonormal basis $\{e_k : k \in \mathbb{Z}^n_+\}$. Following [133], for each $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n_+$ we will write $|k_1, \ldots, k_n\rangle$ for e_k. As was proved by Pusz and Woronowicz [133], there exists a faithful irreducible $*$-representation π of $\operatorname{Pol}_q(\mathbb{C}^n)$ on H

uniquely determined by

$$\pi(x_j)e_k = \sqrt{1-q^2}\sqrt{[k_j+1]_{q^2}}\, q^{\sum_{i>j} k_i} |k_1,\ldots,k_j+1,\ldots,k_n\rangle$$
$$(j=1,\ldots,n,\ k=(k_1,\ldots,k_n)\in\mathbb{Z}_+^n).$$

The completion of $\mathrm{Pol}_q(\mathbb{C}^n)$ with respect to the operator norm $\|a\|_{\mathrm{op}} = \|\pi(a)\|$ is denoted by $C_q(\bar{\mathbb{B}}^n)$ and is called the *algebra of continuous functions on the closed quantum ball* [167]; see also [130, 133].

Observe now that the subalgebra of $\mathrm{Pol}_q(\mathbb{C}^n)$ generated by $x_1,\ldots,x_n$ is exactly $\mathscr{O}_q^{\mathrm{reg}}(\mathbb{C}^n)$. For each $\rho>0$, let γ_ρ be the automorphism of $\mathscr{O}_q^{\mathrm{reg}}(\mathbb{C}^n)$ uniquely determined by $\gamma_\rho(x_i)=\rho x_i$ $(i=1,\ldots,n)$. Define a submultiplicative norm $\|\cdot\|_{\mathbb{B},\rho}^\infty$ on $\mathscr{O}_q^{\mathrm{reg}}(\mathbb{C}^n)$ by

$$\|a\|_{\mathbb{B},\rho}^\infty = \|\gamma_\rho(a)\|_{\mathrm{op}} \qquad (a\in\mathscr{O}_q^{\mathrm{reg}}(\mathbb{C}^n)). \tag{5.2}$$

The completion of $\mathscr{O}_q^{\mathrm{reg}}(\mathbb{C}^n)$ with respect to the family $\{\|\cdot\|_{\mathbb{B},\rho}^\infty : \rho\in(0,r)\}$ of norms will be denoted by $\mathscr{O}_q^{\mathrm{V}}(\mathbb{B}_r^n)$ (the superscript "V" is for "Vaksman"). It follows from the discussion at the beginning of this section that $\mathscr{O}_q^{\mathrm{V}}(\mathbb{B}_r^n)$ is indeed a natural q-analog of $\mathscr{O}(\mathbb{B}_r^n)$.

The main result of this section is the following theorem.

Theorem 5.1. *For each $q\in(0,1)$ and each $r\in(0,+\infty]$, there exists a topological algebra isomorphism*

$$\mathscr{O}_q^{\mathrm{V}}(\mathbb{B}_r^n) \xrightarrow{\sim} \mathscr{O}_q(\mathbb{B}_r^n), \quad x_i\mapsto x_i \quad (i=1,\ldots,n).$$

The proof of Theorem 5.1 will be divided into several lemmas.

Lemma 5.2. *For each $k\in\mathbb{Z}_+^n$, we have*

$$\pi(x^k)e_0 = \sqrt{[k]_{q^2}!}\,(1-q^2)^{\frac{|k|}{2}} w_q(k) e_k.$$

Proof. We use induction on $|k|$. For $|k|=0$ there is nothing to prove. Suppose now that $|k|>0$, and let $m=\min\{i=1,\ldots,n : k_i\neq 0\}$. We have

$$x^k = x_m x^\ell, \quad \text{where} \quad \ell=(0,\ldots,0,k_m-1,k_{m+1},\ldots,k_n).$$

Using the induction hypothesis, we obtain

$$\begin{aligned}
\pi(x^k)e_0 &= \sqrt{[\ell]_{q^2}!}\,(1-q^2)^{\frac{|k|-1}{2}} q^{\sum_{m+1\le i<j} k_ik_j + (k_m-1)\sum_{j>m} k_j}\, \pi(x_m)e_\ell\\
&= \sqrt{[\ell]_{q^2}!}\,(1-q^2)^{\frac{|k|-1}{2}} q^{\sum_{m+1\le i<j} k_ik_j + (k_m-1)\sum_{j>m} k_j} \sqrt{1-q^2}\sqrt{[k_m]_{q^2}}\, q^{\sum_{j>m} k_j}\, e_k\\
&= \sqrt{[k]_{q^2}!}\,(1-q^2)^{\frac{|k|}{2}} q^{\sum_{i<j} k_ik_j}\, e_k.
\end{aligned}$$

□

It will be convenient to introduce one more family of norms on $\mathscr{O}_q^{\mathrm{reg}}(\mathbb{C}^n)$. Namely, for each $\rho>0$ we let

$$\|a\|_{\mathbb{D},\rho}^{(2)} = \Big(\sum_{k\in\mathbb{Z}_+^n} |c_k|^2 w_q^2(k)\rho^{2|k|}\Big)^{1/2} \qquad \Big(a=\sum_{k\in\mathbb{Z}_+^n} c_k x^k \in \mathscr{O}_q^{\mathrm{reg}}(\mathbb{C}^n)\Big).$$

Lemma 5.3. *For each $0<\rho<\tau<+\infty$ we have*

$$\|\cdot\|_{\mathbb{D},\rho}^{(2)} \le \|\cdot\|_{\mathbb{D},\rho} \le \Big(\frac{\tau^2}{\tau^2-\rho^2}\Big)^{n/2} \|\cdot\|_{\mathbb{D},\tau}^{(2)}.$$

Proof. This is a special case of Lemma 2.1. □

Lemma 5.4. *For each $\rho > 0$, we have*

$$\|\cdot\|^{\infty}_{\mathbb{B},\rho} \le \|\cdot\|_{\mathbb{D},\rho}\,, \quad \|\cdot\|^{\infty}_{\mathbb{B},\rho} \ge (q^2;q^2)^{n/2}_{\infty}\|\cdot\|^{(2)}_{\mathbb{D},\rho}. \tag{5.3}$$

Proof. Let us first consider the case $\rho = 1$, so that $\|\cdot\|^{\infty}_{\mathbb{B},1} = \|\cdot\|_{\mathrm{op}}$. By [133], we have[2] $\|x_i\|_{\mathrm{op}} \le 1$ for all $i = 1,\dots,n$. Using the maximality property of $\|\cdot\|_{\mathbb{D},1}$ (see [106, Lemma 5.10]), we conclude that $\|\cdot\|_{\mathrm{op}} \le \|\cdot\|_{\mathbb{D},1}$.

Now take any $a = \sum_k c_k x^k \in \mathscr{O}^{\mathrm{reg}}_q(\mathbb{C}^n)$. Using Lemma 5.2, we see that

$$\|a\|^2_{\mathrm{op}} \ge \|\pi(a)e_0\|^2 = \sum_{k\in\mathbb{Z}^n_+} |c_k|^2 [k]_{q^2}!\,(1-q^2)^{|k|} w^2_q(k). \tag{5.4}$$

Observe that for each $\ell \in \mathbb{N}$

$$[\ell]_{q^2}!\,(1-q^2)^{\ell} = \prod_{j=1}^{\ell}(1-q^{2j}) \ge \prod_{j=1}^{\infty}(1-q^{2j}) = (q^2;q^2)_{\infty},$$

and so for each $k = (k_1,\dots,k_n) \in \mathbb{Z}^n_+$

$$[k]_{q^2}!\,(1-q^2)^{|k|} = \prod_{i=1}^{n}[k_i]_{q^2}!\,(1-q^2)^{k_i} \ge (q^2;q^2)^n_{\infty}.$$

Now it follows from (5.4) that

$$\|a\|^2_{\mathrm{op}} \ge (q^2;q^2)^n_{\infty} \sum_{k\in\mathbb{Z}^n_+} |c_k|^2 w^2_q(k) = (q^2;q^2)^n_{\infty}(\|a\|^{(2)}_{\mathbb{D},1})^2.$$

Thus we have proved (5.3) for $\rho = 1$. The general case reduces to the case $\rho = 1$ by using (5.2) and by observing that

$$\|a\|_{\mathbb{D},\rho} = \|\gamma_\rho(a)\|_{\mathbb{D},1}\,, \quad \|a\|^{(2)}_{\mathbb{D},\rho} = \|\gamma_\rho(a)\|^{(2)}_{\mathbb{D},1}.$$ □

Proof of Theorem 5.1. Applying Lemmas 5.3 and 5.4, we see that the families

$$\{\|\cdot\|_{\mathbb{D},\rho} : \rho \in (0,r)\}, \quad \{\|\cdot\|^{(2)}_{\mathbb{D},\rho} : \rho \in (0,r)\}, \quad \{\|\cdot\|^{\infty}_{\mathbb{B},\rho} : \rho \in (0,r)\}$$

of norms on $\mathscr{O}^{\mathrm{reg}}_q(\mathbb{C}^n)$ are equivalent. The rest follows from Theorem 4.2. □

Remark 5.5. In the special case where $r = \infty$, Theorem 5.1 is equivalent to [106, Theorem 5.16].

Remark 5.6. Lemmas 5.3 and 5.4 imply that for each $0 < \rho < \tau < +\infty$ we have

$$\Big(\frac{\tau^2-\rho^2}{\tau^2}\,(q^2;q^2)_{\infty}\Big)^{n/2}\|\cdot\|_{\mathbb{D},\rho} \le \|\cdot\|^{\infty}_{\mathbb{B},\tau} \le \|\cdot\|_{\mathbb{D},\tau}. \tag{5.5}$$

While the second inequality in (5.5) holds in the classical case $q = 1$ as well, the first inequality in (5.5) becomes useless (since $(q^2;q^2)_{\infty} \to 0$ as $q \to 1$). Geometrically, this can be explained as follows. If we fix τ and take $\rho < \tau$ close enough to τ, then the polydisk of radius ρ will not be contained in the ball of radius τ, and so the supremum over the polydisk (which is less than or equal to $\|\cdot\|_{\mathbb{D},\rho}$) will not be dominated by the supremum $\|\cdot\|^{\infty}_{\mathbb{B},\tau}$ over the ball.

[2] In fact, it is easy to show that $\|x_i\|_{\mathrm{op}} = 1$ for all $i = 1,\dots,n$, but we will not use this equality here.

6. Quantum polydisk as a quotient of the free polydisk

We begin this section by recalling some results from [110]. Let $(A_i)_{i\in I}$ be a family of Arens-Michael algebras. The *analytic free product* [34, 110] of $(A_i)_{i\in I}$ is the coproduct of $(A_i)_{i\in I}$ in the category of Arens-Michael algebras, i.e., an Arens-Michael algebra $\widehat{*}_{i\in I} A_i$ together with a natural isomorphism

$$\mathrm{Hom}_{\mathsf{AM}}(\mathop{\widehat{*}}_{i\in I} A_i, B) \cong \prod_{i\in I} \mathrm{Hom}_{\mathsf{AM}}(A_i, B) \qquad (B \in \mathsf{AM}).$$

The analytic free product always exists and can be constructed explicitly [34,110]. Clearly, the analytic free product is unique up to a unique topological algebra isomorphism over the A_i's.

Let $r > 0$, and let $\mathbb{D}_r = \mathbb{D}_r^1$ denote the open disk of radius r.

Definition 6.1 ([110]). The *algebra of holomorphic functions on the free n-dimensional polydisk of radius r is*

$$\mathscr{F}(\mathbb{D}_r^n) = \mathscr{O}(\mathbb{D}_r) \mathbin{\widehat{*}} \cdots \mathbin{\widehat{*}} \mathscr{O}(\mathbb{D}_r). \tag{6.1}$$

Note that replacing in (6.1) the analytic free product $\widehat{*}$ by the projective tensor product $\widehat{\otimes}$ yields the algebra of holomorphic functions on $\mathbb{D}_r^n$. If $r = \infty$, then $\mathscr{F}(\mathbb{D}_r^n) = \mathscr{F}(\mathbb{C}^n) = \mathscr{F}_n$, the Arens-Michael envelope (2.4) of F_n [110, Proposition 3.9].

The algebra $\mathscr{F}(\mathbb{D}_r^n)$ can also be described more explicitly as follows. For each $i = 1, \ldots, n$, let ζ_i denote the canonical image of the complex coordinate $z \in \mathscr{O}(\mathbb{D}_r)$ under the embedding of the ith factor $\mathscr{O}(\mathbb{D}_r)$ into $\mathscr{F}(\mathbb{D}_r^n)$. Given $d \geq 2$ and $\alpha = (\alpha_1, \ldots, \alpha_d) \in W_n$, let $\mathrm{s}(\alpha)$ denote the cardinality of the set

$$\bigl\{ i \in \{1, \ldots, d-1\} : \alpha_i \neq \alpha_{i+1} \bigr\}.$$

If $|\alpha| \in \{0, 1\}$, we let $\mathrm{s}(\alpha) = |\alpha| - 1$. The next result is [110, Proposition 3.3].

Proposition 6.2. *We have*

$$\mathscr{F}(\mathbb{D}_r^n) = \Bigl\{ a = \sum_{\alpha\in W_n} c_\alpha \zeta_\alpha : \|a\|_{\rho,\tau} = \sum_{\alpha\in W_n} |c_\alpha| \rho^{|\alpha|} \tau^{\mathrm{s}(\alpha)+1} < \infty \; \forall \rho \in (0, r), \; \forall \tau \geq 1 \Bigr\}. \tag{6.2}$$

The topology on $\mathscr{F}(\mathbb{D}_r^n)$ is given by the norms $\|\cdot\|_{\rho,\tau}$ $(0 < \rho < r, \; \tau \geq 1)$, and the multiplication is given by concatenation.

The following universal property of $\mathscr{F}(\mathbb{D}_r^n)$ was proved in [110, Proposition 3.2]. Given an algebra A and an element $a \in A$, the spectrum of a in A will be denoted by $\sigma_A(a)$.

Proposition 6.3. *Let A be an Arens-Michael algebra, and let $a = (a_1, \ldots, a_n)$ be an n-tuple in A^n such that $\sigma_A(a_i) \subset \mathbb{D}_r$ for all $i = 1, \ldots, n$. Then there exists a unique continuous homomorphism $\gamma_a \colon \mathscr{F}(\mathbb{D}_r^n) \to A$ such that $\gamma_a(\zeta_i) = a_i$ for all $i = 1, \ldots, n$. Moreover, the assignment $a \mapsto \gamma_a$ determines a natural isomorphism*

$$\mathrm{Hom}_{\mathsf{AM}}(\mathscr{F}(\mathbb{D}_r^n), A) \cong \{ a \in A^n : \sigma_A(a_i) \subset \mathbb{D}_r \; \forall i = 1, \ldots, n \} \qquad (A \in \mathsf{AM}).$$

Another algebra closely related to $\mathscr{F}(\mathbb{D}_n^r)$ was introduced by J. L. Taylor [160, 161]. We will define it in a slightly more general context. For a Banach space E, the *analytic tensor algebra* $\widehat{T}(E)$ ([34]; cf. also [106, 171, 172]) is given by

$$\widehat{T}(E) = \Bigl\{ a = \sum_{d=0}^{\infty} a_d : a_d \in E^{\widehat{\otimes} d}, \; \|a\|_\rho = \sum_d \|a_d\| \rho^d < \infty \; \forall \rho > 0 \Bigr\},$$

where $E^{\widehat{\otimes} d} = E \mathbin{\widehat{\otimes}} \cdots \mathbin{\widehat{\otimes}} E$ is the dth completed projective tensor power of E. The topology on $\widehat{T}(E)$ is given by the norms $\|\cdot\|_\rho$ ($\rho > 0$), and the multiplication on $\widehat{T}(E)$ is given by concatenation, like on the usual tensor algebra $T(E)$. Each norm $\|\cdot\|_\rho$ is easily seen to be submultiplicative, and so $\widehat{T}(E)$ is an Arens-Michael algebra containing $T(E)$ as a dense subalgebra. As was observed by J. Cuntz [34], $\widehat{T}(E)$ has the universal property that, for every Arens-Michael algebra A, each continuous linear map $E \to A$ uniquely extends to a continuous homomorphism $\widehat{T}(E) \to A$. In other words, there is a natural isomorphism

$$\operatorname{Hom}_{\mathsf{AM}}(\widehat{T}(E), A) \cong \mathscr{L}(E, A) \qquad (A \in \mathsf{AM}),$$

where $\mathscr{L}(E, A)$ is the space of all continuous linear maps from E to A. Note that $\widehat{T}(E)$ was originally defined in the more general setting where E is a complete locally convex space [34], but this generality is not needed here.

Fix now $r > 0$, and let

$$\widehat{T}_r(E) = \Bigl\{ a = \sum_{d=0}^{\infty} a_d : a_d \in E^{\widehat{\otimes} d},\ \|a\|_\rho = \sum_d \|a_d\| \rho^d < \infty \ \forall \rho \in (0, r) \Bigr\}.$$

It follows from the above discussion that $\widehat{T}_r(E)$ is an Arens-Michael algebra containing $T(E)$ as a dense subalgebra. Note that $\widehat{T}_r(E)$ essentially depends on the fixed norm on E (in contrast to $\widehat{T}(E)$, which depends only on the topology of E).

Definition 6.4. Let $\mathbb{C}^n_1$ be the vector space $\mathbb{C}^n$ endowed with the ℓ^1-norm $\|x\| = \sum_{i=1}^n |x_i|$ (where $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$). The algebra $\widehat{T}_r(\mathbb{C}^n_1)$ will be denoted by $\mathscr{F}^{\mathrm{T}}(\mathbb{D}^n_r)$ and will be called *Taylor's algebra of holomorphic functions on the free n-dimensional polydisk of radius r.*

Using the canonical isometric isomorphisms $\mathbb{C}^m_1 \mathbin{\widehat{\otimes}} \mathbb{C}^n_1 \cong \mathbb{C}^{mn}_1$, we see that

$$\mathscr{F}^{\mathrm{T}}(\mathbb{D}^n_r) = \Bigl\{ a = \sum_{\alpha \in W_n} c_\alpha \zeta_\alpha : \|a\|_\rho = \sum_{\alpha \in W_n} |c_\alpha| \rho^{|\alpha|} < \infty \ \forall \rho \in (0, r) \Bigr\}. \tag{6.3}$$

The algebra $\mathscr{F}^{\mathrm{T}}(\mathbb{D}^n_r)$ was introduced by J. L. Taylor [160, 161] and was denoted by $S(r)$ in [160] and by $\mathscr{F}_n(r)$ in [161]. Our notation hints that both $\mathscr{F}(\mathbb{D}^n_r)$ and $\mathscr{F}^{\mathrm{T}}(\mathbb{D}^n_r)$ are natural candidates for the algebra of holomorphic functions on the free polydisk; the superscript "T" is for "Taylor".

Comparing (6.3) with (2.4), we see that $\mathscr{F}^{\mathrm{T}}(\mathbb{D}^n_\infty) = \mathscr{F}_n$. It is also immediate from (6.2) and (6.3) that $\mathscr{F}(\mathbb{D}^n_r) \subset \mathscr{F}^{\mathrm{T}}(\mathbb{D}^n_r)$, and that the embedding $\mathscr{F}(\mathbb{D}^n_r) \to \mathscr{F}^{\mathrm{T}}(\mathbb{D}^n_r)$ is continuous. However, it is easy to observe that $\mathscr{F}(\mathbb{D}^n_r) \neq \mathscr{F}^{\mathrm{T}}(\mathbb{D}^n_r)$ unless $r = \infty$ or $n = 1$; for instance, the element $\sum_k r^{-2k} (\zeta_1 \zeta_2)^k$ belongs to $\mathscr{F}^{\mathrm{T}}(\mathbb{D}^n_r)$, but does not belong to $\mathscr{F}(\mathbb{D}^n_r)$. Moreover, $\mathscr{F}(\mathbb{D}^n_r)$ is nuclear as a locally convex space [110], while $\mathscr{F}^{\mathrm{T}}(\mathbb{D}^n_r)$ is not [84, 160], so they are not isomorphic even as locally convex spaces.

Our next goal is to show that $\mathscr{F}^{\mathrm{T}}(\mathbb{D}^n_r)$ has a remarkable universal property similar in spirit to Proposition 6.3.

Definition 6.5. Let A be an Arens-Michael algebra, and let $r > 0$. We say that an n-tuple $a \in A^n$ is *strictly spectrally r-contractive* if, for each Banach algebra B and each continuous homomorphism $\varphi \colon A \to B$, we have $r_\infty(\varphi(a)) < r$.

Remark 6.6. Observe that, if A and B are Arens-Michael algebras, $\psi\colon A \to B$ is a continuous homomorphism, and $a \in A^n$ is strictly spectrally r-contractive, then so is $\psi(a) \in B^n$.

An equivalent but more handy definition is as follows.

Proposition 6.7. *Let A be an Arens-Michael algebra, and let $\{\|\cdot\|_\lambda : \lambda \in \Lambda\}$ be a directed defining family of submultiplicative seminorms on A. For each $\lambda \in \Lambda$, let A_λ denote the completion of A with respect to $\|\cdot\|_\lambda$. Given $a \in A^n$, let a_λ denote the canonical image of a in A_λ^n. Then the following conditions are equivalent:*

(i) *a is strictly spectrally r-contractive;*
(ii) *$r_\infty(a_\lambda) < r$ for all $\lambda \in \Lambda$;*
(iii) *$\lim_{d\to\infty}\bigl(\sup_{\alpha\in W_{n,d}} \|a_\alpha\|_\lambda\bigr)^{1/d} < r$ for all $\lambda \in \Lambda$.*

Proof. (i) $\Longrightarrow$ (ii) $\iff$ (iii). This is clear.

(ii) $\Longrightarrow$ (i). Let B be a Banach algebra, and let $\varphi\colon A \to B$ be a continuous homomorphism. There exist $\lambda \in \Lambda$ and $C > 0$ such that for all $a \in A$ we have $\|\varphi(a)\| \le C\|a\|_\lambda$. Hence there exists a unique continuous homomorphism $\psi\colon A_\lambda \to B$ such that $\varphi = \psi\tau_\lambda$, where $\tau_\lambda\colon A \to A_\lambda$ is the canonical map. We have

$$r_\infty(\varphi(a)) = r_\infty(\psi(a_\lambda)) \le r_\infty(a_\lambda) < r,$$

and so a is strictly spectrally r-contractive. ☐

Corollary 6.8. *If A is a Banach algebra, then $a \in A^n$ is strictly spectrally r-contractive if and only if $r_\infty(a) < r$.*

Example 6.9. Let $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathscr{F}^{\mathrm{T}}(\mathbb{D}_r^n)^n$. For each $\rho \in (0, r)$, each $d \in \mathbb{Z}_+$, and each $\alpha \in W_{n,d}$, we have $\|\zeta_\alpha\|_\rho^{1/d} = \rho$. Hence $\zeta \in \mathscr{F}^{\mathrm{T}}(\mathbb{D}_r^n)^n$ is strictly spectrally r-contractive, but is not strictly spectrally r'-contractive whenever $r' < r$. The same assertion holds for the n-tuple $z = (z_1, \ldots, z_n) \in \mathscr{O}(\mathbb{D}_r^n)^n$ of coordinate functions on $\mathbb{D}_r^n$. Note that such a phenomenon can never happen in a Banach algebra (see Corollary 6.8).

Example 6.10. Let $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathscr{F}(\mathbb{D}_r^n)^n$, where $n \ge 2$. For each $\rho \in (0, r)$, each $\tau \ge 1$, each $d \in \mathbb{Z}_+$, and each $\alpha \in W_{n,d}$, we have $\|\zeta_\alpha\|_{\rho,\tau}^{1/d} = \rho\tau^{(s(\alpha)+1)/d}$. In particular, for $\alpha = (1, 2, \ldots, 1, 2) \in W_{n,2d}$ we have $\|\zeta_\alpha\|_{\rho,\tau}^{1/2d} = \rho\tau$. Hence $\zeta \in \mathscr{F}(\mathbb{D}_r^n)^n$ is not strictly spectrally R-contractive for any $R > 0$.

Proposition 6.11. *Let A be an Arens-Michael algebra, and let $a = (a_1, \ldots, a_n)$ be a strictly spectrally r-contractive n-tuple in A^n. Then there exists a unique continuous homomorphism $\gamma_a\colon \mathscr{F}^{\mathrm{T}}(\mathbb{D}_r^n) \to A$ such that $\gamma_a(\zeta_i) = a_i$ for all $i = 1, \ldots, n$. Moreover, the assignment $a \mapsto \gamma_a$ determines a natural isomorphism*

$$\mathrm{Hom}_{\mathsf{AM}}(\mathscr{F}^{\mathrm{T}}(\mathbb{D}_r^n), A) \cong \{a \in A^n : a \text{ is str. spec. } r\text{-contractive}\} \qquad (A \in \mathsf{AM}). \tag{6.4}$$

Proof. Let $\|\cdot\|$ be a continuous submultiplicative seminorm on A. Using Proposition 6.7, we can choose $\rho > 0$ such that

$$\lim_{d\to\infty}\Bigl(\sup_{\alpha\in W_{n,d}} \|a_\alpha\|\Bigr)^{1/d} < \rho < r.$$

Hence there exists $d_0 \in \mathbb{Z}_+$ such that

$$\sup_{\alpha\in W_{n,d}} \|a_\alpha\| < \rho^d \qquad (d \ge d_0). \tag{6.5}$$

Choose now $C \geq 1$ such that $\|a_\alpha\| \leq C\rho^{|\alpha|}$ whenever $|\alpha| < d_0$. Together with (6.5), this yields $\|a_\alpha\| \leq C\rho^{|\alpha|}$ for all $\alpha \in W_n$. Hence for each $f = \sum_\alpha c_\alpha \zeta_\alpha \in \mathscr{F}^{\mathrm{T}}(\mathbb{D}_r^n)$ we obtain

$$\sum_{\alpha \in W_n} |c_\alpha| \|a_\alpha\| \leq C \sum_{\alpha \in W_n} |c_\alpha| \rho^{|\alpha|} = C\|f\|_\rho.$$

Therefore the series $\sum_\alpha c_\alpha a_\alpha$ absolutely converges in A, and the mapping

$$\gamma_a \colon \mathscr{F}^{\mathrm{T}}(\mathbb{D}_r^n) \to A, \qquad \sum_{\alpha \in W_n} c_\alpha \zeta_\alpha \mapsto \sum_{\alpha \in W_n} c_\alpha a_\alpha$$

is the required continuous homomorphism. The uniqueness of γ_a is immediate from the density of the free algebra F_n in $\mathscr{F}^{\mathrm{T}}(\mathbb{D}_n^r)$.

Conversely, using Example 6.9 and Remark 6.6, we see that, for each Arens-Michael algebra A and each continuous homomorphism $\varphi \colon \mathscr{F}^{\mathrm{T}}(\mathbb{D}_r^n) \to A$, the n-tuple $\varphi(\zeta) \in A^n$ is strictly spectrally r-contractive. Thus (6.4) is indeed a bijection, as required. $\square$

Remark 6.12. Proposition 6.11 can easily be extended to the algebra $\widehat{T}_r(E)$ for any Banach space E. Now the set of strictly spectrally r-contractive n-tuples on the right-hand side of (6.4) should be replaced by the set of all $\psi \in \mathscr{L}(E, A)$ such that the image of the unit ball of E under ψ is a strictly spectrally r-contractive set in A. Related results can be found in [43].

Theorem 6.13. *Let $q \in \mathbb{C}^\times$, $n \in \mathbb{N}$, and $r \in (0, +\infty]$.*

(i) *There exists a surjective continuous homomorphism*
$$\pi \colon \mathscr{F}(\mathbb{D}_r^n) \to \mathscr{O}_q(\mathbb{D}_r^n), \qquad \zeta_i \mapsto x_i \quad (i = 1, \ldots, n). \tag{6.6}$$
(ii) $\operatorname{Ker} \pi$ *coincides with the closed two-sided ideal of $\mathscr{F}(\mathbb{D}_r^n)$ generated by the elements $\zeta_i \zeta_j - q\zeta_j \zeta_i$ $(i, j = 1, \ldots, n,\ i < j)$.*
(iii) $\operatorname{Ker} \pi$ *is a complemented subspace of $\mathscr{F}(\mathbb{D}_r^n)$.*
(iv) *Under the identification $\mathscr{O}_q(\mathbb{D}_r^n) \cong \mathscr{F}(\mathbb{D}_r^n)/\operatorname{Ker} \pi$, the norm $\|\cdot\|_{\mathbb{D},\rho}$ on $\mathscr{O}_q(\mathbb{D}_r^n)$ is equal to the quotient of the norm $\|\cdot\|_{\rho,\tau}$ on $\mathscr{F}(\mathbb{D}_r^n)$ $(\rho \in (0, r),\ \tau \geq 1)$.*

Parts (i)–(iii) of the above theorem were proved in [110, Theorem 7.13] in the more general multiparameter case. The proof of the "quantitative" part (iv) will be given below, together with the proof of Theorem 6.14.

Theorem 6.14. *Let $q \in \mathbb{C}^\times$, $n \in \mathbb{N}$, and $r \in (0, +\infty]$.*

(i) *There exists a surjective continuous homomorphism*
$$\pi^{\mathrm{T}} \colon \mathscr{F}^{\mathrm{T}}(\mathbb{D}_r^n) \to \mathscr{O}_q(\mathbb{D}_r^n), \qquad \zeta_i \mapsto x_i \quad (i = 1, \ldots, n). \tag{6.7}$$
(ii) $\operatorname{Ker} \pi^{\mathrm{T}}$ *coincides with the closed two-sided ideal of $\mathscr{F}^{\mathrm{T}}(\mathbb{D}_r^n)$ generated by the elements $\zeta_i \zeta_j - q\zeta_j \zeta_i$ $(i, j = 1, \ldots, n,\ i < j)$.*
(iii) $\operatorname{Ker} \pi^{\mathrm{T}}$ *is a complemented subspace of $\mathscr{F}^{\mathrm{T}}(\mathbb{D}_r^n)$.*
(iv) *Under the identification $\mathscr{O}_q(\mathbb{D}_r^n) \cong \mathscr{F}^{\mathrm{T}}(\mathbb{D}_r^n)/\operatorname{Ker} \pi^{\mathrm{T}}$, the norm $\|\cdot\|_{\mathbb{D},\rho}$ on $\mathscr{O}_q(\mathbb{D}_r^n)$ is equal to the quotient of the norm $\|\cdot\|_\rho$ on $\mathscr{F}^{\mathrm{T}}(\mathbb{D}_r^n)$ $(\rho \in (0, r))$.*

To prove Theorem 6.14, we need some notation from [106]. For each $d \in \mathbb{Z}_+$, let the symmetric group S_d act on $W_{n,d}$ via $\sigma(\alpha) = \alpha\sigma^{-1}$ $(\alpha \in W_{n,d},\ \sigma \in S_d)$. Clearly, for each $\alpha \in W_{n,d}$ and $\sigma \in S_d$ there exists a unique $\lambda(\sigma, \alpha) \in \mathbb{C}^\times$ such that

$$x_\alpha = \lambda(\sigma, \alpha) x_{\sigma(\alpha)}.$$

Given $k = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n$ with $|k| = d$, let

$$\delta(k) = (\underbrace{1, \ldots, 1}_{k_1}, \ldots, \underbrace{n, \ldots, n}_{k_n}) \in W_{n,d}.$$

By [106, Proposition 5.12], we have

$$w_q(k) = \min\{|\lambda(\sigma, \delta(k))| : \sigma \in S_d\}. \tag{6.8}$$

Clearly, for each $\alpha \in \mathrm{p}^{-1}(k)$ (where p is given by (4.5)) there exists $\sigma_\alpha \in S_d$ such that $\alpha = \sigma_\alpha(\delta(k))$. Therefore $x^k = \lambda(\sigma_\alpha, \delta(k)) x_\alpha$. Comparing with (4.7), we see that

$$\lambda(\sigma_\alpha, \delta(k)) = \mathrm{t}(\alpha)^{-1}. \tag{6.9}$$

Observe also that, if $\alpha \in W_{n,d}$ and $\sigma_1, \sigma_2 \in S_d$ are such that $\sigma_1(\alpha) = \sigma_2(\alpha)$, then $\lambda(\sigma_1, \alpha) = \lambda(\sigma_2, \alpha)$. In other words, $\lambda(\sigma, \alpha)$ depends only on α and $\sigma(\alpha)$. Since the orbit of $\delta(k)$ under the action of S_d is exactly $\mathrm{p}^{-1}(k)$, we conclude from (6.8) and (6.9) that

$$w_q(k) = \min\{|\mathrm{t}(\alpha)|^{-1} : \alpha \in \mathrm{p}^{-1}(k)\}. \tag{6.10}$$

Proof of Theorem 6.14. Let $x = (x_1, \ldots, x_n) \in \mathscr{O}_q(\mathbb{D}_r^n)^n$. For each $\rho \in (0, r)$, each $d \in \mathbb{Z}_+$, and each $\alpha \in W_{n,d}$, we clearly have $\|x_\alpha\|_{\mathbb{D},\rho} \le \rho^d$. Hence x is strictly spectrally r-contractive, and Proposition 6.11 yields the required homomorphism (6.7). The homomorphism π given by (6.6) is then the composition of π^{T} with the canonical embedding

$$\nu \colon \mathscr{F}(\mathbb{D}_r^n) \to \mathscr{F}^{\mathrm{T}}(\mathbb{D}_r^n), \qquad \zeta_i \mapsto \zeta_i \quad (i = 1, \ldots, n).$$

By Theorem 6.13, π is onto, and hence so is π^{T}. This proves (i).

Since $\operatorname{Ker}\pi$ is a complemented subspace of $\mathscr{F}(\mathbb{D}_r^n)$, there exists a continuous linear map $\varkappa \colon \mathscr{O}_q(\mathbb{D}_r^n) \to \mathscr{F}(\mathbb{D}_r^n)$ such that $\pi\varkappa = 1$. Letting $\varkappa^{\mathrm{T}} = \nu\varkappa \colon \mathscr{O}_q(\mathbb{D}_r^n) \to \mathscr{F}^{\mathrm{T}}(\mathbb{D}_r^n)$, we see that $\pi^{\mathrm{T}}\varkappa^{\mathrm{T}} = 1$, whence $\operatorname{Ker}\pi^{\mathrm{T}}$ is a complemented subspace of $\mathscr{F}^{\mathrm{T}}(\mathbb{D}_r^n)$. This proves (iii).

Using the density of $\operatorname{Im}\nu$ in $\mathscr{F}^{\mathrm{T}}(\mathbb{D}_r^n)$, we obtain

$$\operatorname{Ker}\pi^{\mathrm{T}} = \operatorname{Im}(1 - \varkappa^{\mathrm{T}}\pi^{\mathrm{T}}) = \overline{\operatorname{Im}\big((1 - \varkappa^{\mathrm{T}}\pi^{\mathrm{T}})\nu\big)} = \overline{\operatorname{Im}\big(\nu(1 - \varkappa\pi)\big)} = \overline{\nu(\operatorname{Ker}\pi)}.$$

Now (ii) follows from Theorem 6.13 (ii).

Since (6.6) and (6.7) are surjective, the Open Mapping Theorem yields topological isomorphisms

$$\mathscr{O}_q(\mathbb{D}_r^n) \cong \mathscr{F}(\mathbb{D}_r^n)/\operatorname{Ker}\pi \cong \mathscr{F}^{\mathrm{T}}(\mathbb{D}_r^n)/\operatorname{Ker}\pi^{\mathrm{T}}.$$

To prove parts (iv) of Theorems 6.13 and 6.14, we have to show that

$$\|\cdot\|_{\mathbb{D},\rho} = \|\cdot\|_\rho^\wedge = \|\cdot\|_{\rho,\tau}^\wedge, \tag{6.11}$$

where $\|\cdot\|_\rho^\wedge$ and $\|\cdot\|_{\rho,\tau}^\wedge$ are the quotient norms of $\|\cdot\|_\rho$ and $\|\cdot\|_{\rho,\tau}$, respectively. Let $f = \sum_{\alpha \in W_n} c_\alpha \zeta_\alpha \in \mathscr{F}^{\mathrm{T}}(\mathbb{D}_r^n)$. We have

$$\pi^{\mathrm{T}}(f) = \sum_{\alpha \in W_n} c_\alpha x_\alpha = \sum_{k \in \mathbb{Z}_+^n} \Big(\sum_{\alpha \in \mathrm{p}^{-1}(k)} c_\alpha \mathrm{t}(\alpha)\Big) x^k.$$

Together with (6.10), this yields

$$\begin{aligned}\|\pi^{\mathrm{T}}(f)\|_{\mathbb{D},\rho} &\le \sum_{k\in\mathbb{Z}_+^n}\Bigl(\sum_{\alpha\in\mathrm{p}^{-1}(k)}|c_\alpha \mathrm{t}(\alpha)|\Bigr)\|x^k\|_{\mathbb{D},\rho}\\ &\le \sum_{k\in\mathbb{Z}_+^n}\Bigl(\max_{\alpha\in\mathrm{p}^{-1}(k)}|\mathrm{t}(\alpha)|\sum_{\alpha\in\mathrm{p}^{-1}(k)}|c_\alpha|\Bigr)w_q(k)\rho^{|k|}\\ &= \sum_{k\in\mathbb{Z}_+^n}\Bigl(\sum_{\alpha\in\mathrm{p}^{-1}(k)}|c_\alpha|\Bigr)\rho^{|k|} = \|f\|_\rho.\end{aligned} \tag{6.12}$$

If now $f\in\mathscr{F}(\mathbb{D}_r^n)$ and $\tau\ge 1$, then

$$\|\pi(f)\|_{\mathbb{D},\rho} = \|\pi^{\mathrm{T}}(\nu(f))\|_{\mathbb{D},\rho} \le \|\nu(f)\|_\rho = \|f\|_{\rho,1} \le \|f\|_{\rho,\tau}. \tag{6.13}$$

From (6.12) and (6.13), we conclude that $\|\cdot\|_{\mathbb{D},\rho}\le\|\cdot\|^\wedge_\rho$ and $\|\cdot\|_{\mathbb{D},\rho}\le\|\cdot\|^\wedge_{\rho,\tau}$. On the other hand, both $\|\cdot\|^\wedge_\rho$ and $\|\cdot\|^\wedge_{\rho,\tau}$ are submultiplicative norms on $\mathscr{O}_q(\mathbb{D}_r^n)$, and we have $\|x_i\|^\wedge_\rho \le \|\zeta_i\|_\rho = \rho$ and $\|x_i\|^\wedge_{\rho,\tau}\le\|\zeta_i\|_{\rho,\tau} = \rho$. By the maximality property of $\|\cdot\|_{\mathbb{D},\rho}$ [106, Lemma 5.10], it follows that $\|\cdot\|^\wedge_\rho\le\|\cdot\|_{\mathbb{D},\rho}$ and $\|\cdot\|^\wedge_{\rho,\tau}\le\|\cdot\|_{\mathbb{D},\rho}$. Together with the above estimates, this gives (6.11) and completes the proof of (iv), both for Theorem 6.13 and Theorem 6.14. ☐

7. Quantum ball as a quotient of the free ball

The goal of this section is to prove a quantum ball analog of Theorems 6.13 and 6.14. Towards this goal, it will be convenient to introduce a "hilbertian" version of the algebra $\widehat{T}_r(E)$. In what follows, given Hilbert spaces H_1 and H_2, their Hilbert tensor product will be denoted by $H_1\mathbin{\dot\otimes} H_2$. The Hilbert tensor product of n copies of a Hilbert space H will be denoted by $H^{\dot\otimes n}$.

Given $r>0$ and a Hilbert space H, let

$$\dot{T}_r(H) = \Bigl\{a = \sum_{d=0}^\infty a_d : a_d\in H^{\dot\otimes d},\ \|a\|^\bullet_\rho = \sum_d \|a_d\|\rho^d < \infty\ \forall\rho\in(0,r)\Bigr\}.$$

Clearly, $\dot{T}_r(H)$ is a Fréchet space with respect to the topology determined by the norms $\|\cdot\|^\bullet_\rho$ ($\rho>0$). Similarly to the case of $\widehat{T}_r(E)$, it is easy to check that each norm $\|\cdot\|^\bullet_\rho$ is submultiplicative on the tensor algebra $T(H)\subset\dot{T}_r(H)$. Therefore there exists a unique continuous multiplication on $\dot{T}_r(H)$ extending that of $T(H)$, and $\dot{T}_r(H)$ becomes an Arens-Michael algebra containing $T(H)$ as a dense subalgebra.

Definition 7.1. Let $\mathbb{C}^n_2$ be the vector space $\mathbb{C}^n$ endowed with the inner product $\langle x,y\rangle = \sum_{i=1}^n x_i\bar{y}_i$. The algebra $\dot{T}_r(\mathbb{C}^n_2)$ will be denoted by $\mathscr{F}(\mathbb{B}^n_r)$ and will be called *the algebra of holomorphic functions on the free n-dimensional ball of radius r.*

Using the canonical isometric isomorphisms $\mathbb{C}^m_2\mathbin{\dot\otimes}\mathbb{C}^n_2\cong\mathbb{C}^{mn}_2$, we see that

$$\mathscr{F}(\mathbb{B}^n_r) = \Bigl\{a = \sum_{\alpha\in W_n} c_\alpha\zeta_\alpha : \|a\|^\bullet_\rho = \sum_{d=0}^\infty\Bigl(\sum_{|\alpha|=d}|c_\alpha|^2\Bigr)^{1/2}\rho^d < \infty\ \forall\rho\in(0,r)\Bigr\}. \tag{7.1}$$

Let us first observe that Definition 7.1 is consistent with our convention that $\mathbb{B}^n_\infty = \mathbb{D}^n_\infty = \mathbb{C}^n$.

Proposition 7.2. *We have $\mathscr{F}(\mathbb{B}^n_\infty)=\mathscr{F}(\mathbb{D}^n_\infty)=\mathscr{F}^{\mathrm{T}}(\mathbb{D}^n_\infty)=\mathscr{F}_n$ as topological algebras.*

Proof. We already know that $\mathscr{F}(\mathbb{D}^n_\infty)=\mathscr{F}^{\mathrm{T}}(\mathbb{D}^n_\infty)=\mathscr{F}_n$ (see Section 6). To complete the proof, it suffices to show that the families

$$\{\|\cdot\|_\rho:\rho>0\}\quad\text{and}\quad\{\|\cdot\|^\bullet_\rho:\rho>0\}$$

of norms (where $\|\cdot\|_\rho$ is given by (6.3)) are equivalent on F_n. Given $a=\sum_\alpha c_\alpha\zeta_\alpha\in F_n$, we clearly have

$$\|a\|_\rho=\sum_d\Bigl(\sum_{|\alpha|=d}|c_\alpha|\Bigr)\rho^d,$$

whence $\|a\|^\bullet_\rho\le\|a\|_\rho$. Conversely, by using the Cauchy-Bunyakowsky-Schwarz inequality, we obtain

$$\|a\|_\rho\le\sum_d\Bigl(\sum_{|\alpha|=d}|c_\alpha|^2\Bigr)^{1/2}n^{d/2}\rho^d=\|a\|^\bullet_{\rho\sqrt{n}}.\qquad\square$$

Our next goal is to show that $\mathscr{F}(\mathbb{B}^n_r)$ coincides with the algebra $Hol(\mathscr{B}(\mathcal{H})^n_r)$ of "free holomorphic functions on the open operatorial ball" introduced by G. Popescu [118]. To this end, let us recall some results from [118].

Let H be a Hilbert space, let $\mathscr{B}(H)$ denote the algebra of bounded linear operators on H, and let $T=(T_1,\ldots,T_n)$ be an n-tuple in $\mathscr{B}(H)^n$. Following [118], we identify T with the "row" operator acting from the Hilbert direct sum $H^n=H\oplus\cdots\oplus H$ to H. Thus we have $\|T\|=\|\sum_{i=1}^n T_iT_i^*\|^{1/2}$.

Let $\mathfrak{F}_n$ denote the algebra of all formal series $f=\sum_{\alpha\in W_n}c_\alpha\zeta_\alpha$ (where $c_\alpha\in\mathbb{C}$) with the obvious multiplication. In other words, $\mathfrak{F}_n=\varprojlim_d F_n/I^d$, where I is the ideal of the free algebra $F_n=\mathbb{C}\langle\zeta_1,\ldots,\zeta_n\rangle$ generated by $\zeta_1,\ldots,\zeta_n$. For each $f=\sum_\alpha c_\alpha\zeta_\alpha\in\mathfrak{F}_n$, the *radius of convergence* $R(f)\in[0,+\infty]$ is given by

$$\frac{1}{R(f)}=\limsup_{d\to\infty}\Bigl(\sum_{|\alpha|=d}|c_\alpha|^2\Bigr)^{\frac{1}{2d}}.$$

By [118, Theorem 1.1], for each $T\in\mathscr{B}(H)^n$ such that $\|T\|<R(f)$, the series

$$\sum_{d=0}^\infty\Bigl(\sum_{|\alpha|=d}c_\alpha T_\alpha\Bigr)\tag{7.2}$$

converges in $\mathscr{B}(H)$ and, moreover, $\sum_d\|\sum_{|\alpha|=d}c_\alpha T_\alpha\|<\infty$. On the other hand, if H is infinite-dimensional, then for each $R'>R(f)$ there exists $T\in\mathscr{B}(H)^n$ with $\|T\|=R'$ such that the series (7.2) diverges. The collection of all $f\in\mathfrak{F}_n$ such that $R(f)\ge r$ is denoted by $Hol(\mathscr{B}(\mathcal{H})^n_r)$. By [118, Theorem 1.4], $Hol(\mathscr{B}(\mathcal{H})^n_r)$ is a subalgebra of $\mathfrak{F}_n$. For each $f\in Hol(\mathscr{B}(\mathcal{H})^n_r)$, each Hilbert space H, and each $T\in\mathscr{B}(H)^n$ with $\|T\|<r$, the sum of the series (7.2) is denoted by $f(T)$. The map

$$\gamma_T\colon\ Hol(\mathscr{B}(\mathcal{H})^n_r)\to\mathscr{B}(H),\qquad f\mapsto f(T),$$

is an algebra homomorphism.

Fix an infinite-dimensional Hilbert space $\mathcal{H}$, and, for each $\rho\in(0,r)$, define a seminorm $\|\cdot\|^P_\rho$ on $Hol(\mathscr{B}(\mathcal{H})^n_r)$ by

$$\|f\|^P_\rho=\sup\{\|f(T)\|:T\in\mathscr{B}(\mathcal{H})^n,\ \|T\|\le\rho\}.$$

By [118, Theorem 5.6], $Hol(\mathscr{B}(\mathcal{H})^n_r)$ is a Fréchet space with respect to the topology determined by the family $\{\|\cdot\|^P_\rho : \rho \in (0,r)\}$ of seminorms.

The following result is implicitly contained in [118]. For the reader's convenience, we give a proof here.

Proposition 7.3. *For each $r \in (0,+\infty]$, $Hol(\mathscr{B}(\mathcal{H})^n_r) = \mathscr{F}(\mathbb{B}^n_r)$ as topological algebras.*

Proof. By Lemma 2.1,

$$\mathscr{F}(\mathbb{B}^n_r) = \Big\{ f = \sum_{\alpha \in W_n} c_\alpha \zeta_\alpha : \|f\|^{(\infty)}_\rho = \sup_{d \in \mathbb{Z}_+} \Big(\sum_{|\alpha|=d} |c_\alpha|^2 \Big)^{1/2} \rho^d < \infty \ \forall \rho \in (0,r) \Big\}.$$

On the other hand, by [118, Corollary 1.2], for each $f = \sum_\alpha c_\alpha \zeta_\alpha \in \mathfrak{F}_n$ we have

$$R(f) = \sup \Big\{ \rho \ge 0 : \text{ the sequence } \Big\{ \Big(\sum_{|\alpha|=d} |c_\alpha|^2 \Big)^{1/2} \rho^d \Big\}_{d \in \mathbb{Z}_+} \text{ is bounded} \Big\}.$$

Thus $Hol(\mathscr{B}(\mathcal{H})^n_r) = \mathscr{F}(\mathbb{B}^n_r)$ as algebras. To complete the proof, it suffices to show that for each $\rho \in (0,r)$ we have

$$\|\cdot\|^{(\infty)}_\rho \le \|\cdot\|^P_\rho \le \|\cdot\|^\bullet_\rho \tag{7.3}$$

on $\mathscr{F}(\mathbb{B}^n_r)$.

Fix $\rho \in (0,r)$ and $T \in \mathscr{B}(\mathcal{H})^n$ such that $\|T\| \le \rho$. For each $f = \sum_\alpha c_\alpha \zeta_\alpha \in \mathscr{F}(\mathbb{B}^n_r)$, we have

$$\begin{aligned}\|f(T)\| &= \Big\| \sum_\alpha c_\alpha T_\alpha \Big\| \le \sum_{d=0}^\infty \Big\| \sum_{|\alpha|=d} c_\alpha T_\alpha \Big\| \le \sum_{d=0}^\infty \Big(\sum_{|\alpha|=d} |c_\alpha|^2 \Big)^{1/2} \Big\| \sum_{|\alpha|=d} T_\alpha T_\alpha^* \Big\|^{1/2} \\ &\le \sum_{d=0}^\infty \Big(\sum_{|\alpha|=d} |c_\alpha|^2 \Big)^{1/2} \Big\| \sum_{i=1}^n T_i T_i^* \Big\|^{d/2} \le \sum_{d=0}^\infty \Big(\sum_{|\alpha|=d} |c_\alpha|^2 \Big)^{1/2} \rho^d = \|f\|^\bullet_\rho.\end{aligned}$$

Thus $\|f\|^P_\rho \le \|f\|^\bullet_\rho$. Let now $S_1, \ldots, S_n$ be the left creation operators on the full Fock space

$$H = \dot{\bigoplus_{d \in \mathbb{Z}_+}} (\mathbb{C}^n_2)^{\dot{\otimes} d} \tag{7.4}$$

(where $\dot{\oplus}$ stands for the Hilbert direct sum). Recall that $S_i x = e_i \otimes x$ for each $x \in (\mathbb{C}^n_2)^{\dot{\otimes} d}$ and each $d \in \mathbb{Z}_+$, where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{C}^n$. Let $e_0 \in H$ be the "vacuum vector", i.e., any element of the 0th direct summand $\mathbb{C}$ in (7.4) with $\|e_0\| = 1$. Then $\{S_\alpha e_0 : \alpha \in W_n\}$ is an orthonormal basis of H. Let now $T_i = \rho S_i$ $(i = 1, \ldots, n)$, and let $T = (T_1, \ldots, T_n)$. We have $\|T\| = \rho \| \sum_{i=1}^n S_i S_i^* \|^{1/2} = \rho$, and

$$\begin{aligned}\|f(T)\| &= \Big\| \sum_\alpha c_\alpha \rho^{|\alpha|} S_\alpha \Big\| \ge \Big\| \sum_\alpha c_\alpha \rho^{|\alpha|} S_\alpha e_0 \Big\| = \Big(\sum_\alpha |c_\alpha|^2 \rho^{2|\alpha|} \Big)^{1/2} \\ &= \Big(\sum_d \Big(\sum_{|\alpha|=d} |c_\alpha|^2 \Big) \rho^{2d} \Big)^{1/2} \ge \sup_d \Big(\sum_{|\alpha|=d} |c_\alpha|^2 \Big)^{1/2} \rho^d = \|f\|^{(\infty)}_\rho.\end{aligned}$$

This yields (7.3) and completes the proof. □

Remark 7.4. It is immediate from (7.3) that each seminorm $\|\cdot\|^P_\rho$ on $Hol(\mathscr{B}(\mathcal{H})^n_r)$ is actually a norm.

Remark 7.5. Note that the proof of Proposition 7.3 does not rely on the completeness of $Hol(\mathscr{B}(\mathcal{H})^n_r)$. Since $\mathscr{F}(\mathbb{B}^n_r)$ is obviously complete, we see that Proposition 7.3 readily implies [118, Theorem 5.6].

For future use, it will be convenient to modify the norms $\|\cdot\|^\bullet_\rho$ on $\mathscr{F}(\mathbb{B}^n_r)$ as follows. Observe that we have a $\mathbb{Z}_+$-grading $F_n = \bigoplus_{d\in\mathbb{Z}_+}(F_n)_d$, where

$$(F_n)_d = \operatorname{span}\{\zeta_\alpha : |\alpha| = d\} = (\mathbb{C}^n_2)^{\dot\otimes d}.$$

The norm $\|\cdot\|^\bullet_\rho$ on F_n is then given by $\|f\|^\bullet_\rho = \sum_{d\in\mathbb{Z}_+}\|f_d\|\rho^d$, where $f_d \in (F_n)_d$ is the dth homogeneous component of f and $\|\cdot\|$ is the hilbertian norm on $(F_n)_d$. Consider now a finer $\mathbb{Z}^n_+$-grading $F_n = \bigoplus_{k\in\mathbb{Z}^n_+}(F_n)_k$, where

$$(F_n)_k = \operatorname{span}\{\zeta_\alpha : \alpha \in \mathrm{p}^{-1}(k)\}.$$

Observe that for each $d \in \mathbb{Z}_+$ we have $(F_n)_d = \bigoplus_{k\in(\mathbb{Z}^n_+)_d}(F_n)_k$. For each $\rho > 0$, define a new norm $\|\cdot\|^\circ_\rho$ on F_n by

$$\|f\|^\circ_\rho = \sum_{k\in\mathbb{Z}^n_+}\|f_k\|\rho^{|k|},$$

where $f_k \in (F_n)_k$ is the kth homogeneous component of f and $\|\cdot\|$ is the hilbertian norm on $(F_n)_k$ inherited from $(F_n)_{|k|}$. Explicitly, for $f = \sum_\alpha c_\alpha\zeta_\alpha \in F_n$ we have

$$\|f\|^\circ_\rho = \sum_{k\in\mathbb{Z}^n_+}\Bigg(\sum_{\alpha\in\mathrm{p}^{-1}(k)}|c_\alpha|^2\Bigg)^{1/2}\rho^{|k|}.$$

Lemma 7.6. *For each $\rho > 0$, the norm $\|\cdot\|^\circ_\rho$ is submultiplicative, and the families*

$$\{\|\cdot\|^\bullet_\rho : \rho \in (0,r)\} \quad \text{and} \quad \{\|\cdot\|^\circ_\rho : \rho \in (0,r)\} \tag{7.5}$$

of norms are equivalent on F_n.

Proof. Let $r, s \in \mathbb{Z}^n_+$, $a \in (F_n)_r \subset (\mathbb{C}^n_2)^{\dot\otimes|r|}$, and $b \in (F_n)_s \subset (\mathbb{C}^n_2)^{\dot\otimes|s|}$. We clearly have $\|ab\| = \|a\otimes b\| = \|a\|\|b\|$. Hence for each $f, g \in F_n$ and each $\rho > 0$ we obtain

$$\begin{aligned}\|fg\|^\circ_\rho &= \sum_{k\in\mathbb{Z}^n_+}\|(fg)_k\|\rho^{|k|} = \sum_{k\in\mathbb{Z}^n_+}\Bigg\|\sum_{r+s=k}f_r g_s\Bigg\|\rho^{|k|}\\ &\le \sum_{k\in\mathbb{Z}^n_+}\sum_{r+s=k}\|f_r\|\|g_s\|\rho^{|k|} = \sum_{r,s\in\mathbb{Z}^n_+}\|f_r\|\|g_s\|\rho^{|r|}\rho^{|s|} = \|f\|^\circ_\rho\|g\|^\circ_\rho.\end{aligned}$$

Thus $\|\cdot\|^\circ_\rho$ is submultiplicative.

Let now $f \in F_n$, and let $\rho \in (0,r)$. By using the decomposition $(F_n)_d = \bigoplus_{k\in(\mathbb{Z}^n_+)_d}(F_n)_k$, we obtain

$$\begin{aligned}\|f\|^\bullet_\rho &= \sum_{d\in\mathbb{Z}_+}\|f_d\|\rho^d = \sum_{d\in\mathbb{Z}_+}\Bigg\|\sum_{k\in(\mathbb{Z}^n_+)_d}f_k\Bigg\|\rho^d\\ &\le \sum_{d\in\mathbb{Z}_+}\sum_{k\in(\mathbb{Z}^n_+)_d}\|f_k\|\rho^d = \sum_{k\in\mathbb{Z}^n_+}\|f_k\|\rho^{|k|} = \|f\|^\circ_\rho.\end{aligned} \tag{7.6}$$

Conversely, the orthogonality of the f_k's and the Cauchy-Bunyakowsky-Schwarz inequality yield the estimate

$$\|f\|_\rho^\circ = \sum_{d\in\mathbb{Z}_+} \sum_{k\in(\mathbb{Z}_+^n)_d} \|f_k\|\rho^d \le \sum_{d\in\mathbb{Z}_+} \Bigl\| \sum_{k\in(\mathbb{Z}_+^n)_d} f_k \Bigr\| \bigl|(\mathbb{Z}_+^n)_d\bigr|^{1/2} \rho^d. \tag{7.7}$$

By (4.10), for each $\rho_1 \in (\rho, r)$ we have $\lim_{d\to\infty} \bigl|(\mathbb{Z}_+^n)_d\bigr|^{1/2} (\rho/\rho_1)^d = 0$. As a consequence,

$$C = \sup_{d\in\mathbb{Z}_+} \bigl|(\mathbb{Z}_+^n)_d\bigr|^{1/2} (\rho/\rho_1)^d < \infty.$$

Together with (7.7), this implies that

$$\|f\|_\rho^\circ \le C \sum_{d\in\mathbb{Z}_+} \Bigl\| \sum_{k\in(\mathbb{Z}_+^n)_d} f_k \Bigr\| \rho_1^d = C \sum_{d\in\mathbb{Z}_+} \|f_d\|\rho_1^d = C\|f\|_{\rho_1}^\bullet. \tag{7.8}$$

Now (7.6) and (7.8) imply that the families (7.5) of norms are equivalent. □

Proposition 7.7. *We have*

$$\mathscr{F}(\mathbb{B}_r^n) = \Bigl\{ a = \sum_{\alpha\in W_n} c_\alpha \zeta_\alpha : \|a\|_\rho^\circ = \sum_{k\in\mathbb{Z}_+^n} \Bigl(\sum_{\alpha\in \mathrm{p}^{-1}(k)} |c_\alpha|^2 \Bigr)^{1/2} \rho^{|k|} < \infty \ \forall \rho\in(0,r) \Bigr\}.$$

Moreover, each norm $\|\cdot\|_\rho^\circ$ on $\mathscr{F}(\mathbb{B}_r^n)$ is submultiplicative.

Proof. Immediate from Lemma 7.6. □

Let us introduce some notation. Given $\alpha = (\alpha_1, \ldots, \alpha_d) \in W_n$, let

$$\mathrm{m}(\alpha) = \bigl|\{(i,j) : 1 \le i < j \le d,\ \alpha_i > \alpha_j\}\bigr|.$$

Lemma 7.8. *For each $\alpha \in W_n$, we have $\mathrm{t}(\alpha) = q^{-\mathrm{m}(\alpha)}$. In other words, $x_\alpha = q^{-\mathrm{m}(\alpha)} x^{\mathrm{p}(\alpha)}$ in $\mathscr{O}_q^{\mathrm{reg}}(\mathbb{C}^n)$.*

Proof. We use induction on $\mathrm{m}(\alpha)$. If $\mathrm{m}(\alpha) = 0$, then $\alpha = \delta(\mathrm{p}(\alpha))$, whence $x_\alpha = x^{\mathrm{p}(\alpha)}$ and $\mathrm{t}(\alpha) = 1$. Suppose now that $\mathrm{m}(\alpha) = r > 0$, and assume that $\mathrm{t}(\beta) = q^{-\mathrm{m}(\beta)}$ for all $\beta \in W_n$ such that $\mathrm{m}(\beta) < r$. Let $s = \min\{i \ge 2 : \alpha_i < \alpha_{i-1}\}$, and let $\beta = (s\, s-1)(\alpha)$. It is elementary to check that $\mathrm{m}(\beta) = r-1$. By the induction hypothesis, we have $x_\beta = q^{1-r} x^k$, where $k = \mathrm{p}(\beta) = \mathrm{p}(\alpha)$. Therefore $x_\alpha = q^{-1} x_\beta = q^{-r} x^k$, i.e., $\mathrm{t}(\alpha) = q^{-r}$, as required. □

Lemma 7.9. *For each $k \in \mathbb{Z}_+^n$, we have*

$$\|x^k\|_{\mathbb{B},1} = \Bigl(\sum_{\alpha\in\mathrm{p}^{-1}(k)} |q|^{-2\mathrm{m}(\alpha)} \Bigr)^{-1/2}.$$

Proof. Given $s \in \mathbb{Z}_+$, let

$$\mathrm{inv}(k,s) = \bigl|\{\alpha \in \mathrm{p}^{-1}(k) : \mathrm{m}(\alpha) = s\}\bigr|.$$

By [5, Theorem 3.6], we have

$$\frac{\bigl[|k|\bigr]_q!}{[k]_q!} = \sum_{s\ge 0} \mathrm{inv}(k,s) q^s = \sum_{\alpha\in\mathrm{p}^{-1}(k)} q^{\mathrm{m}(\alpha)}.$$

Together with Corollary 3.14, this implies that

$$\|x^k\|_{\mathbb{B},1} = \left(\frac{[k]_{|q|^{-2}}!}{\big[|k|\big]_{|q|^{-2}}!}\right)^{1/2} = \left(\sum_{\alpha\in \mathrm{p}^{-1}(k)} |q|^{-2\mathrm{m}(\alpha)}\right)^{-1/2}. \qquad \square$$

We are now ready to prove the main result of this section.

Theorem 7.10. *Let $q \in \mathbb{C}^\times$, $n \in \mathbb{N}$, and $r \in (0, +\infty]$.*

(i) *There exists a surjective continuous homomorphism*

$$\pi\colon \mathscr{F}(\mathbb{B}^n_r) \to \mathscr{O}_q(\mathbb{B}^n_r), \qquad \zeta_i \mapsto x_i \quad (i = 1, \ldots, n).$$

(ii) $\operatorname{Ker}\pi$ *coincides with the closed two-sided ideal of $\mathscr{F}(\mathbb{B}^n_r)$ generated by the elements $\zeta_i\zeta_j - q\zeta_j\zeta_i$ $(i, j = 1, \ldots, n,\ i < j)$.*

(iii) $\operatorname{Ker}\pi$ *is a complemented subspace of $\mathscr{F}(\mathbb{B}^n_r)$.*

(iv) *Under the identification $\mathscr{O}_q(\mathbb{B}^n_r) \cong \mathscr{F}(\mathbb{B}^n_r)/\operatorname{Ker}\pi$, the norm $\|\cdot\|_{\mathbb{B},\rho}$ on $\mathscr{O}_q(\mathbb{D}^n_r)$ is equal to the quotient of the norm $\|\cdot\|^\circ_\rho$ on $\mathscr{F}(\mathbb{B}^n_r)$ $(\rho \in (0, r))$.*

Proof. Let $f = \sum_{\alpha\in W_n} c_\alpha\zeta_\alpha \in \mathscr{F}(\mathbb{B}^n_r)$. We claim that the family

$$\left\{\sum_{\alpha\in\mathrm{p}^{-1}(k)} c_\alpha x_\alpha : k \in \mathbb{Z}^n_+\right\}$$

is absolutely summable in $\mathscr{O}_q(\mathbb{B}^n_r)$. Indeed, using Lemma 7.8, the Cauchy-Bunyakowsky-Schwarz inequality, and Lemma 7.9, for each $\rho \in (0, r)$ we obtain

$$\begin{aligned}
\sum_{k\in\mathbb{Z}^n_+}\Bigg\|\sum_{\alpha\in\mathrm{p}^{-1}(k)} c_\alpha x_\alpha\Bigg\|_{\mathbb{B},\rho} &= \sum_{k\in\mathbb{Z}^n_+}\Bigg|\sum_{\alpha\in\mathrm{p}^{-1}(k)} c_\alpha q^{-\mathrm{m}(\alpha)}\Bigg|\|x^k\|_{\mathbb{B},\rho}\\
&\le \sum_{k\in\mathbb{Z}^n_+}\left(\sum_{\alpha\in\mathrm{p}^{-1}(k)}|c_\alpha|^2\right)^{1/2}\left(\sum_{\alpha\in\mathrm{p}^{-1}(k)}|q^{-2\mathrm{m}(\alpha)}|\right)^{1/2}\|x^k\|_{\mathbb{B},1}\rho^{|k|}\\
&= \sum_{k\in\mathbb{Z}^n_+}\left(\sum_{\alpha\in\mathrm{p}^{-1}(k)}|c_\alpha|^2\right)^{1/2}\rho^{|k|} = \|f\|^\circ_\rho. \qquad (7.9)
\end{aligned}$$

Hence there exists a continuous linear map

$$\pi\colon \mathscr{F}(\mathbb{B}^n_r) \to \mathscr{O}_q(\mathbb{B}^n_r), \quad \sum_{\alpha\in W_n} c_\alpha\zeta_\alpha \mapsto \sum_{k\in\mathbb{Z}^n_+}\left(\sum_{\alpha\in\mathrm{p}^{-1}(k)} c_\alpha x_\alpha\right).$$

Clearly, π is an algebra homomorphism. Moreover, (7.9) implies that

$$\|\pi(f)\|_{\mathbb{B},\rho} \le \|f\|^\circ_\rho \qquad (f \in \mathscr{F}(\mathbb{B}^n_r)). \qquad (7.10)$$

Let us now construct a continuous linear map $\varkappa\colon \mathscr{O}_q(\mathbb{B}^n_r) \to \mathscr{F}(\mathbb{B}^n_r)$ such that $\pi\varkappa = 1$. To this end, observe that for each $k \in \mathbb{Z}^n_+$ we have

$$x^k = \pi\left(\sum_{\alpha\in\mathrm{p}^{-1}(k)} c_\alpha q^{\mathrm{m}(\alpha)}\zeta_\alpha\right) \qquad (7.11)$$

as soon as

$$\sum_{\alpha\in\mathrm{p}^{-1}(k)} c_\alpha = 1. \qquad (7.12)$$

We have

$$\Big\| \sum_{\alpha\in \mathrm{p}^{-1}(k)} c_\alpha q^{\mathrm{m}(\alpha)} \zeta_\alpha \Big\|_\rho^\circ = \Big(\sum_{\alpha\in \mathrm{p}^{-1}(k)} |c_\alpha|^2 |q|^{2\mathrm{m}(\alpha)} \Big)^{1/2} \rho^{|k|}. \tag{7.13}$$

Our strategy is to minimize (7.13) under the condition (7.12). First observe that replacing c_α by $\operatorname{Re} c_\alpha$ preserves (7.12) and does not increase (7.13). Thus we may assume that $c_\alpha \in \mathbb{R}$. An elementary computation involving Lagrange multipliers shows that the minimum of (7.13) under the condition (7.12) is attained at

$$c_\alpha^0 = |q|^{-2\mathrm{m}(\alpha)} \Big(\sum_{\beta\in \mathrm{p}^{-1}(k)} |q|^{-2\mathrm{m}(\beta)} \Big)^{-1}$$

and is equal to

$$\Big(\sum_{\beta\in \mathrm{p}^{-1}(k)} |q|^{-2\mathrm{m}(\beta)} \Big)^{-1/2} \rho^{|k|} = \|x^k\|_{\mathbb{B},\rho} \tag{7.14}$$

(see Lemma 7.9). Let now

$$a_k = \sum_{\alpha\in \mathrm{p}^{-1}(k)} c_\alpha^0 q^{\mathrm{m}(\alpha)} \zeta_\alpha.$$

By (7.11) and (7.14), we have

$$\pi(a_k) = x^k, \quad \|a_k\|_\rho^\circ = \|x^k\|_{\mathbb{B},\rho}. \tag{7.15}$$

Let us now define

$$\varkappa \colon \mathscr{O}_q(\mathbb{B}_r^n) \to \mathscr{F}(\mathbb{B}_r^n), \quad \sum_{k\in\mathbb{Z}_+^n} c_k x^k \mapsto \sum_{k\in\mathbb{Z}_+^n} c_k a_k.$$

By (7.15), we have

$$\sum_{k\in\mathbb{Z}_+^n} |c_k| \|a_k\|_\rho^\circ = \sum_{k\in\mathbb{Z}_+^n} |c_k| \|x^k\|_{\mathbb{B},\rho} = \Big\| \sum_{k\in\mathbb{Z}_+^n} c_k x^k \Big\|_{\mathbb{B},\rho},$$

whence $\varkappa$ is indeed a continuous linear map from $\mathscr{O}_q(\mathbb{B}_r^n)$ to $\mathscr{F}(\mathbb{B}_r^n)$. Moreover,

$$\|\varkappa(f)\|_\rho^\circ \le \|f\|_{\mathbb{B},\rho} \qquad (f \in \mathscr{O}_q(\mathbb{B}_r^n)). \tag{7.16}$$

By (7.15), we also have $\pi\varkappa = 1$. This proves (i) and (iii).

Let now $I \subset \mathscr{F}(\mathbb{B}_r^n)$ denote the closed two-sided ideal generated by $\zeta_i\zeta_j - q\zeta_j\zeta_i$ $(i<j)$. Clearly, $I \subset \operatorname{Ker}\pi$, and hence π induces a continuous homomorphism

$$\bar\pi \colon \mathscr{F}(\mathbb{B}_r^n)/I \to \mathscr{O}_q(\mathbb{B}_r^n), \quad \bar\zeta_i \mapsto x_i \quad (i=1,\ldots,n),$$

where we let $\bar f = f + I \in \mathscr{F}(\mathbb{B}_r^n)/I$ for each $f \in \mathscr{F}(\mathbb{B}_r^n)$. Let $\bar\varkappa \colon \mathscr{O}_q(\mathbb{B}_r^n) \to \mathscr{F}(\mathbb{B}_r^n)/I$ denote the composition of $\varkappa$ with the quotient map $\mathscr{F}(\mathbb{B}_r^n) \to \mathscr{F}(\mathbb{B}_r^n)/I$. It is immediate from $\pi\varkappa = 1$ that $\bar\pi\bar\varkappa = 1$. On the other hand, we obviously have $\bar\varkappa(x_i) = \bar\zeta_i$ for each i, and it follows from the definition of I that $\bar\varkappa$ is an algebra homomorphism. Since the elements $\bar\zeta_1,\ldots,\bar\zeta_n$ generate a dense subalgebra of $\mathscr{F}(\mathbb{B}_r^n)/I$, we conclude that $\operatorname{Im}\bar\varkappa$ is dense in $\mathscr{F}(\mathbb{B}_r^n)/I$. Together with $\bar\pi\bar\varkappa = 1$, this implies that $\bar\pi$ and $\bar\varkappa$ are topological isomorphisms. Therefore $I = \operatorname{Ker}\pi$, which proves (ii).

To prove (iv), let us identify $\mathscr{F}(\mathbb{B}_r^n)/I$ with $\mathscr{O}_q(\mathbb{B}_r^n)$ via $\bar\pi$, and let $\|\cdot\|_\rho^\wedge$ denote the quotient norm of $\|\cdot\|_\rho^\circ$ $(\rho\in(0,r))$. By (7.10), we have $\|\cdot\|_{\mathbb{B},\rho} \le \|\cdot\|_\rho^\wedge$, while (7.16) together with $\pi\varkappa = 1$ yields the opposite estimate. This completes the proof. ☐

8. Deformations

In this section, we explain in which sense the algebras $\mathscr{O}_q(\mathbb{D}^n_r)$ and $\mathscr{O}_q(\mathbb{B}^n_r)$ are deformations of $\mathscr{O}(\mathbb{D}^n_r)$ and $\mathscr{O}(\mathbb{B}^n_r)$, respectively. For basic facts on locally convex bundles and for related notation we refer to Appendix A.

8.1. **Construction of $\mathscr{O}_{\mathrm{def}}(\mathbb{D}^n_r)$ and $\mathscr{O}_{\mathrm{def}}(\mathbb{B}^n_r)$.** If K is a commutative Fréchet algebra, then by a *Fréchet K-algebra* we mean a complete metrizable locally convex K-algebra A such that the action $K \times A \to A$, $(k, a) \mapsto ka$, is continuous. Given a reduced Stein space X and a Fréchet algebra A, we let $\mathscr{O}(X, A)$ denote the Fréchet algebra of all holomorphic A-valued functions on X. Clearly, $\mathscr{O}(X, A)$ is a Fréchet $\mathscr{O}(X)$-algebra in a canonical way. By [58, Chap. II, §3, no. 3] (see also [59, Chap. II, Theorem 4.14]), we have a topological isomorphism

$$\mathscr{O}(X) \mathbin{\widehat{\otimes}} A \to \mathscr{O}(X, A), \qquad f \otimes a \mapsto (x \mapsto f(x)a).$$

From now on, we identify $\mathscr{O}(X)$ and A with subalgebras of $\mathscr{O}(X, A)$ via the embeddings $f \mapsto f \otimes 1$ and $a \mapsto 1 \otimes a$, respectively.

Let $z \in \mathscr{O}(\mathbb{C}^\times)$ denote the complex coordinate. Fix $r \in (0, +\infty]$, and let $I_{\mathbb{D}}$, $I^{\mathrm{T}}_{\mathbb{D}}$, and $I_{\mathbb{B}}$ denote the closed two-sided ideals of $\mathscr{O}(\mathbb{C}^\times, \mathscr{F}(\mathbb{D}^n_r))$, $\mathscr{O}(\mathbb{C}^\times, \mathscr{F}^{\mathrm{T}}(\mathbb{D}^n_r))$, and $\mathscr{O}(\mathbb{C}^\times, \mathscr{F}(\mathbb{B}^n_r))$, respectively, generated by the elements $\zeta_i\zeta_j - z\zeta_j\zeta_i$ $(i < j)$. Consider the Fréchet $\mathscr{O}(\mathbb{C}^\times)$-algebras

$$\begin{aligned}
\mathscr{O}_{\mathrm{def}}(\mathbb{D}^n_r) &= \mathscr{O}(\mathbb{C}^\times, \mathscr{F}(\mathbb{D}^n_r))/I_{\mathbb{D}},\\
\mathscr{O}^{\mathrm{T}}_{\mathrm{def}}(\mathbb{D}^n_r) &= \mathscr{O}(\mathbb{C}^\times, \mathscr{F}^{\mathrm{T}}(\mathbb{D}^n_r))/I^{\mathrm{T}}_{\mathbb{D}},\\
\mathscr{O}_{\mathrm{def}}(\mathbb{B}^n_r) &= \mathscr{O}(\mathbb{C}^\times, \mathscr{F}(\mathbb{B}^n_r))/I_{\mathbb{B}}.
\end{aligned} \tag{8.1}$$

We will use the following simplified notation for the respective Fréchet algebra bundles:

$$\mathsf{E}(\mathbb{D}^n_r) = \mathsf{E}(\mathscr{O}_{\mathrm{def}}(\mathbb{D}^n_r)), \quad \mathsf{E}^{\mathrm{T}}(\mathbb{D}^n_r) = \mathsf{E}(\mathscr{O}^{\mathrm{T}}_{\mathrm{def}}(\mathbb{D}^n_r)), \quad \mathsf{E}(\mathbb{B}^n_r) = \mathsf{E}(\mathscr{O}_{\mathrm{def}}(\mathbb{B}^n_r)).$$

Our goal is to show that the fibers of $\mathscr{O}_{\mathrm{def}}(\mathbb{D}^n_r)$ and $\mathscr{O}^{\mathrm{T}}_{\mathrm{def}}(\mathbb{D}^n_r)$ over $q \in \mathbb{C}^\times$ are isomorphic to $\mathscr{O}_q(\mathbb{D}^n_r)$, while the fiber of $\mathscr{O}_{\mathrm{def}}(\mathbb{B}^n_r)$ over $q \in \mathbb{C}^\times$ is isomorphic to $\mathscr{O}_q(\mathbb{B}^n_r)$.

Lemma 8.1. *Let $\mathfrak{A}$ be a Fréchet algebra, and let $I, J \subset \mathfrak{A}$ be closed two-sided ideals. Denote by $q_I \colon \mathfrak{A} \to \mathfrak{A}/I$ and $q_J \colon \mathfrak{A} \to \mathfrak{A}/J$ the quotient maps, and let $I_0 = \overline{q_J(I)}$, $J_0 = \overline{q_I(J)}$. Then there exist topological algebra isomorphisms*

$$\mathfrak{A}/(\overline{I + J}) \cong (\mathfrak{A}/I)/J_0 \cong (\mathfrak{A}/J)/I_0 \tag{8.2}$$

induced by the identity map on $\mathfrak{A}$.

Proof. Elementary. $\square$

Lemma 8.2. *Let F be a Fréchet algebra, X be a reduced Stein space, $I \subset \mathscr{O}(X, F)$ be a closed two-sided ideal, and $A = \mathscr{O}(X, F)/I$. For each $x \in X$, let $\varepsilon^F_x \colon \mathscr{O}(X, F) \to F$ denote the evaluation map at x, and let $I_x = \overline{\varepsilon^F_x(I)}$. Then there exists a Fréchet algebra*

isomorphism $A_x \cong F/I_x$ *making the diagram*

$$\begin{tikzcd} & \mathscr{O}(X,F) \arrow[dl, "\mathrm{quot}"'] \arrow[dr, "\varepsilon_x^F"] & \\ A \arrow[d, "\mathrm{quot}"'] & & F \arrow[d, "\mathrm{quot}"] \\ A_x \arrow[rr, "\sim"] & & F/I_x \end{tikzcd} \tag{8.3}$$

commute (here quot *are the respective quotient maps).*

Proof. Given $x \in X$, let $\mathfrak{m}_x = \{f \in \mathscr{O}(X) : f(x) = 0\}$. We claim that ε_x^F is onto, and that $\operatorname{Ker} \varepsilon_x^F = \overline{\mathfrak{m}_x \mathscr{O}(X,F)}$. Indeed, by applying the functor $(-) \widehat{\otimes} F$ to the exact sequence

$$0 \to \mathfrak{m}_x \xrightarrow{i_x} \mathscr{O}(X) \xrightarrow{\varepsilon_x} \mathbb{C} \to 0$$

and by using the canonical isomorphism $\mathscr{O}(X) \widehat{\otimes} F \cong \mathscr{O}(X,F)$, we obtain

$$0 \to \mathfrak{m}_x \widehat{\otimes} F \xrightarrow{i_x \otimes 1_F} \mathscr{O}(X,F) \xrightarrow{\varepsilon_x^F} F \to 0. \tag{8.4}$$

Since $(-) \widehat{\otimes} F$ is cokernel preserving [107, Prop. 3.3], it follows that ε_x^F is onto and that $\operatorname{Im}(i_x \otimes 1_F)$ is dense[1] in $\operatorname{Ker} \varepsilon_x^F$. This implies that $\operatorname{Ker} \varepsilon_x^F \subset \overline{\mathfrak{m}_x \mathscr{O}(X,F)}$. The reverse inclusion is obvious. Thus $\operatorname{Ker} \varepsilon_x^F = \overline{\mathfrak{m}_x \mathscr{O}(X,F)}$, as claimed.

Let now $\mathfrak{A} = \mathscr{O}(X,F)$ and $J = \overline{\mathfrak{m}_x \mathfrak{A}}$. It follows from the above discussion and from the Open Mapping Theorem that we can identify F with $\mathfrak{A}/J$. Under this identification, ε_x^F becomes the quotient map $q_J \colon \mathfrak{A} \to \mathfrak{A}/J$. Observe also that $\overline{\mathfrak{m}_x A} = \overline{q_I(J)}$, where $q_I \colon \mathfrak{A} \to \mathfrak{A}/I = A$ is the quotient map. Now Lemma 8.1 implies that

$$F/I_x \cong (\mathfrak{A}/J)/\overline{q_J(I)} \cong (\mathfrak{A}/I)/\overline{q_I(J)} = A/\overline{\mathfrak{m}_x A} = A_x.$$

The commutativity of (8.3) is clear from the construction. $\square$

Theorem 8.3. *For each $q \in \mathbb{C}^\times$, we have topological algebra isomorphisms*

$$\mathscr{O}_{\mathrm{def}}(\mathbb{D}_r^n)_q \cong \mathsf{E}(\mathbb{D}_r^n)_q \cong \mathscr{O}_q(\mathbb{D}_r^n), \tag{8.5}$$

$$\mathscr{O}_{\mathrm{def}}^{\mathrm{T}}(\mathbb{D}_r^n)_q \cong \mathsf{E}^{\mathrm{T}}(\mathbb{D}_r^n)_q \cong \mathscr{O}_q(\mathbb{D}_r^n), \tag{8.6}$$

$$\mathscr{O}_{\mathrm{def}}(\mathbb{B}_r^n)_q \cong \mathsf{E}(\mathbb{B}_r^n)_q \cong \mathscr{O}_q(\mathbb{B}_r^n). \tag{8.7}$$

Proof. Applying Lemma 8.2 with $F = \mathscr{F}(\mathbb{D}_r^n)$ and $I = I_{\mathbb{D}}$, we see that for each $q \in \mathbb{C}^\times$ there exists a topological algebra isomorphism $\mathscr{O}_{\mathrm{def}}(\mathbb{D}_r^n)_q \cong F/I_q$, where $I_q = \overline{\varepsilon_q^F(I)}$. Observe that I_q is precisely the closed two-sided ideal of F generated by the elements $\zeta_i\zeta_j - q\zeta_j\zeta_i$ ($i < j$). Now (8.5) follows from Theorem 6.13. Similarly, letting $F = \mathscr{F}^{\mathrm{T}}(\mathbb{D}_r^n)$ and $I = I_{\mathbb{D}}^{\mathrm{T}}$, and using Theorem 6.14 instead of Theorem 6.13, we obtain (8.6). Finally, letting $F = \mathscr{F}(\mathbb{B}_r^n)$ and $I = I_{\mathbb{B}}$, and using Theorem 7.10 instead of Theorem 6.13, we obtain (8.7). $\square$

Our next goal is to show that the Fréchet $\mathscr{O}(\mathbb{C}^\times)$-algebras $\mathscr{O}_{\mathrm{def}}(\mathbb{D}_r^n)$ and $\mathscr{O}_{\mathrm{def}}^{\mathrm{T}}(\mathbb{D}_r^n)$ not only have isomorphic fibers (see (8.5) and (8.6)), but are in fact isomorphic. Towards this goal, we need some notation and several lemmas. For brevity, we denote the elements $\zeta_i + I_{\mathbb{D}}$ and $z + I_{\mathbb{D}}$ of $\mathscr{O}_{\mathrm{def}}(\mathbb{D}_r^n)$ by x_i and z, respectively. The same convention applies

[1] In fact, the nuclearity of $\mathscr{O}(X)$ implies that (8.4) is exact [159, Prop. 4.2], but this stronger property is not needed for our purposes.

to $\mathscr{O}_{\mathrm{def}}^{\mathrm{T}}(\mathbb{D}_r^n)$ and $\mathscr{O}_{\mathrm{def}}(\mathbb{B}_r^n)$. By looking at (6.2), (6.3), and (7.1), we see that $\mathscr{F}(\mathbb{D}_r^n) \subset \mathscr{F}^{\mathrm{T}}(\mathbb{D}_r^n) \subset \mathscr{F}(\mathbb{B}_r^n)$; moreover, both inclusions are continuous. These inclusions induce Fréchet $\mathscr{O}(\mathbb{C}^\times)$-algebra homomorphisms

$$\mathscr{O}_{\mathrm{def}}(\mathbb{D}_r^n) \xrightarrow{i_{\mathbb{D}\mathbb{D}}} \mathscr{O}_{\mathrm{def}}^{\mathrm{T}}(\mathbb{D}_r^n) \xrightarrow{i_{\mathbb{D}\mathbb{B}}} \mathscr{O}_{\mathrm{def}}(\mathbb{B}_r^n) \tag{8.8}$$

taking each x_i to x_i and z to z.

Let $\mathscr{O}_{\mathrm{def}}^{\mathrm{reg}}(\mathbb{C}^n)$ denote the algebra generated by $n+2$ elements $x_1, \ldots, x_n, z, z^{-1}$ subject to the relations

$$\begin{aligned} x_i x_j &= z x_j x_i && (1 \le i < j \le n), \\ x_i z &= z x_i && (1 \le i \le n); \\ z z^{-1} &= z^{-1} z = 1. \end{aligned}$$

Observe that $\mathscr{O}_{\mathrm{def}}^{\mathrm{reg}}(\mathbb{C}^n)$ is exactly the algebra R given by (1.1). Clearly, there are algebra homomorphisms from $\mathscr{O}_{\mathrm{def}}^{\mathrm{reg}}(\mathbb{C}^n)$ to each of the algebras $\mathscr{O}_{\mathrm{def}}(\mathbb{D}_r^n)$, $\mathscr{O}_{\mathrm{def}}^{\mathrm{T}}(\mathbb{D}_r^n)$, and $\mathscr{O}_{\mathrm{def}}(\mathbb{B}_r^n)$, taking each x_i to x_i and z to z. Together with (8.8), these homomorphisms fit into the commutative diagram

$$\begin{tikzcd} \mathscr{O}_{\mathrm{def}}(\mathbb{D}_r^n) \arrow[r, "i_{\mathbb{D}\mathbb{D}}"] & \mathscr{O}_{\mathrm{def}}^{\mathrm{T}}(\mathbb{D}_r^n) \arrow[r, "i_{\mathbb{D}\mathbb{B}}"] & \mathscr{O}_{\mathrm{def}}(\mathbb{B}_r^n) \\ & \mathscr{O}_{\mathrm{def}}^{\mathrm{reg}}(\mathbb{C}^n) \arrow[ul, "i_{\mathbb{D}}"] \arrow[u, "i_{\mathbb{D}}^{\mathrm{T}}"] \arrow[ur, "i_{\mathbb{B}}"'] & \end{tikzcd} \tag{8.9}$$

Lemma 8.4. (i) *The set $\{x^k z^p : k \in \mathbb{Z}_+^n,\ p \in \mathbb{Z}\}$ is a vector space basis of $\mathscr{O}_{\mathrm{def}}^{\mathrm{reg}}(\mathbb{C}^n)$.*
(ii) *The homomorphisms $i_{\mathbb{D}}$, $i_{\mathbb{D}}^{\mathrm{T}}$, and $i_{\mathbb{B}}$ are injective and have dense images.*

Proof. Obviously, $\mathscr{O}_{\mathrm{def}}^{\mathrm{reg}}(\mathbb{C}^n)$ is spanned by $\{x^k z^p : k \in \mathbb{Z}_+^n,\ p \in \mathbb{Z}\}$. Let $a = \sum_{k,p} c_{kp} x^k z^p \in \mathscr{O}_{\mathrm{def}}^{\mathrm{reg}}(\mathbb{C}^n)$, and suppose that $i_{\mathbb{B}}(a) = 0$. Then for each $q \in \mathbb{C}^\times$ we have $i_{\mathbb{B}}(a)_q = 0$. Identifying $\mathscr{O}_{\mathrm{def}}(\mathbb{B}_r^n)_q$ with $\mathscr{O}_q(\mathbb{B}_r^n)$ (see (8.7)), we see that $\sum_k (\sum_p c_{kp} q^p) x^k = 0$ in $\mathscr{O}_q(\mathbb{B}_r^n)$. Hence $\sum_p c_{kp} q^p = 0$ for each k. Since this holds for every $q \in \mathbb{C}^\times$, we conclude that $c_{kp} = 0$ for all k and p. This implies that the set $\{x^k z^p : k \in \mathbb{Z}_+^n,\ p \in \mathbb{Z}\}$ is linearly independent in $\mathscr{O}_{\mathrm{def}}^{\mathrm{reg}}(\mathbb{C}^n)$, and that $\operatorname{Ker} i_{\mathbb{B}} = 0$. By looking at (8.9), we see that $\operatorname{Ker} i_{\mathbb{D}}^{\mathrm{T}} = \operatorname{Ker} i_{\mathbb{D}} = 0$. The rest is clear. □

Lemma 8.5. *For each $k \in \mathbb{Z}_+^n$ and each integer $m \in [0, \sum_{i<j} k_i k_j]$ there exists $\alpha \in W_n$ such that $\mathrm{p}(\alpha) = k$, $\mathrm{s}(\alpha) \le n+2$, and $x^k = x_\alpha z^m$ in $\mathscr{O}_{\mathrm{def}}^{\mathrm{reg}}(\mathbb{C}^n)$.*

Proof. Given $d \in \mathbb{Z}_+$, let $Z_d = \{\zeta_\alpha : \alpha \in W_{n,d}\} \subset F_n$. Clearly, $\alpha \mapsto \zeta_\alpha$ is a 1-1 correspondence between $W_{n,d}$ and Z_d. Hence we have an action of S_d on Z_d given by $\sigma(\zeta_\alpha) = \zeta_{\sigma(\alpha)}$ ($\alpha \in W_{n,d}$, $\sigma \in S_d$).

We now present an explicit procedure of constructing α out of k. Let $d = |k|$, $\beta = \delta(k)$, and $w_0 = \zeta^k = \zeta_\beta$. Let also $T = \{(i\ \ i+1) : 1 \le i \le d-1\} \subset S_d$. By interchanging the first letter β_1 of w_0 with the subsequent letters, we obtain $d-1$ elements of T such that the action of their composition on w_0 yields

$$\zeta_{\beta_2} \cdots \zeta_{\beta_d} \zeta_{\beta_1}. \tag{8.10}$$

Next, by interchanging the first letter β_2 of (8.10) with the subsequent letters (except for the last one), we obtain $d-2$ elements of T such that the action of their composition on

(8.10) yields

$$\zeta_{\beta_3}\cdots\zeta_{\beta_d}\zeta_{\beta_2}\zeta_{\beta_1}.$$

Continuing this procedure, after finitely many steps we obtain $\sigma_1,\ldots,\sigma_r\in T$ such that

$$(\sigma_r\cdots\sigma_1)(w_0)=\zeta_{\beta_d}\cdots\zeta_{\beta_1}=\zeta_n^{k_n}\cdots\zeta_1^{k_1}.$$

Let $w_i=(\sigma_i\cdots\sigma_1)(w_0)$ $(i=1,\ldots,r)$, and let $\pi\colon F_n\to\mathscr{O}_{\mathrm{def}}^{\mathrm{reg}}(\mathbb{C}^n)$ denote the homomorphism taking each ζ_i to x_i. It follows from the above construction that for each $i=0,\ldots,r-1$ we have either $\pi(w_i)=\pi(w_{i+1})z$ or $\pi(w_i)=\pi(w_{i+1})$. We also have

$$\pi(w_0)=x^k=x_n^{k_n}\cdots x_1^{k_1}z^{\sum_{i<j}k_ik_j}=\pi(w_r)z^{\sum_{i<j}k_ik_j}.$$

Hence there exists $t\in\{0,\ldots,r\}$ such that $\pi(w_0)=\pi(w_t)z^m$. In other words, if $\alpha\in W_{n,d}$ is such that $w_t=\zeta_\alpha$, then we have $x^k=x_\alpha z^m$, as required. Since $\mathrm{p}^{-1}(k)$ is invariant under the action of S_d, we have $\mathrm{p}(\alpha)=k$.

To complete the proof, we have to show that $\mathrm{s}(\alpha)\le n+2$. It follows from the construction that w_t is of the form

$$\begin{gathered}\zeta_r^{k_r-\ell}\zeta_{r+1}^{k_{r+1}}\cdots\zeta_{s-1}^{k_{s-1}}\zeta_s^p\zeta_r\zeta_s^{k_s-p}\zeta_{s+1}^{k_{s+1}}\cdots\zeta_n^{k_n}\zeta_r^{\ell-1}\zeta_{r-1}^{k_{r-1}}\cdots\zeta_1^{k_1}\\(1\le r\le n-1,\quad 1\le\ell\le k_r,\quad r+1\le s\le n,\quad 0\le p\le k_s)\end{gathered}\tag{8.11}$$

(here r is the number of the letter that is currently moving from left to right, $\ell-1$ is the number of instances of ζ_r that are already moved to their final destination, s shows the position where the moving letter is located, and p shows how many copies of ζ_s are already interchanged with the moving letter). An elementary computation shows that for each word ζ_α of the form (8.11) we have $\mathrm{s}(\alpha)\le n+2$. □

Lemma 8.6. *For each $n\in\mathbb{Z}_+$ and each $\alpha\in W_n$, we have* $\mathrm{m}(\alpha)\le\sum_{i<j}\mathrm{p}_i(\alpha)\mathrm{p}_j(\alpha)$.

Proof. We use induction on n. For $n=0$, there is nothing to prove. Suppose now that $n\ge 1$, and that the assertion holds for all words in W_{n-1}. Given $r\in\mathbb{Z}_+$, let $\bar n_r=(n,\ldots,n)\in W_{n,r}$ (r copies of n). If $\alpha=\bar n_{|\alpha|}$, then there is nothing to prove. Assume that $\alpha\ne\bar n_{|\alpha|}$, and represent α as

$$\alpha=\bar n_{r_1}\beta_1\bar n_{r_2}\beta_2\ldots\bar n_{r_k}\beta_k\bar n_{r_{k+1}},$$

where $r_1,\ldots,r_{k+1}\ge 0$, $\beta_i\in W_{n-1}$, $|\beta_i|>0$ for all $i=1,\ldots,k$. Let $\beta=\beta_1\ldots\beta_k\in W_{n-1}$. We have

$$\mathrm{m}(\alpha)=\mathrm{m}(\beta)+r_1(|\beta_1|+\cdots+|\beta_k|)+r_2(|\beta_2|+\cdots+|\beta_k|)+\cdots+r_k|\beta_k|.\tag{8.12}$$

On the other hand, $\mathrm{p}_i(\alpha)=\mathrm{p}_i(\beta)$ for all $i\le n-1$, and $\mathrm{p}_n(\alpha)=r_1+\cdots+r_{k+1}$. Together with (8.12) and the induction hypothesis, this yields

$$\begin{aligned}\mathrm{m}(\alpha)&\le\mathrm{m}(\beta)+(r_1+\cdots+r_k)|\beta|\le\sum_{i<j<n}\mathrm{p}_i(\beta)\mathrm{p}_j(\beta)+\mathrm{p}_n(\alpha)|\beta|\\&=\sum_{i<j<n}\mathrm{p}_i(\alpha)\mathrm{p}_j(\alpha)+\mathrm{p}_n(\alpha)\sum_{i=1}^{n-1}\mathrm{p}_i(\alpha)=\sum_{i<j}\mathrm{p}_i(\alpha)\mathrm{p}_j(\alpha).\end{aligned}$$

□

Given $f\in\mathscr{O}(\mathbb{C}^\times)$ and $t\ge 1$, let

$$\|f\|_t=\sum_{n\in\mathbb{Z}}|c_n(f)|t^{|n|},$$

where $c_n(f)$ is the nth Laurent coefficient of f at 0. We will use the simple fact that the family $\{\|\cdot\|_t : t \ge 1\}$ of norms determines the standard (i.e., compact-open) topology on $\mathscr{O}(\mathbb{C}^\times)$.

Lemma 8.7. *Given $\rho \in (0, r)$ and $\tau, t \ge 1$, let $\|\cdot\|_{\rho,\tau,t}$ denote the quotient seminorm on $\mathscr{O}_{\mathrm{def}}(\mathbb{D}^n_r)$ associated to the projective tensor norm $\|\cdot\|_t \otimes_\pi \|\cdot\|_{\rho,\tau}$ on $\mathscr{O}(\mathbb{C}^\times) \mathbin{\widehat{\otimes}} \mathscr{F}(\mathbb{D}^n_r)$. We have*

$$\lim_{d\to\infty} \Bigl(\sup_{\alpha \in W_{n,d}} \|x_\alpha\|_{\rho,\tau,t}\Bigr)^{1/d} \le \rho. \tag{8.13}$$

Proof. Let $\alpha \in W_{n,d}$. Repeating the argument of Lemma 7.8, we see that $x_\alpha = x^{\mathrm{p}(\alpha)} z^{-\mathrm{m}(\alpha)}$ in $\mathscr{O}^{\mathrm{reg}}_{\mathrm{def}}(\mathbb{C}^n)$ (and hence in $\mathscr{O}_{\mathrm{def}}(\mathbb{D}^n_r)$). By Lemma 8.6, the n-tuple $k = \mathrm{p}(\alpha)$ and the integer $m = \mathrm{m}(\alpha)$ satisfy the conditions of Lemma 8.5. Hence there exists $\beta \in W_{n,d}$ such that $x^{\mathrm{p}(\alpha)} = x_\beta z^{\mathrm{m}(\alpha)}$ and $\mathrm{s}(\beta) \le n+2$. Thus we have $x_\alpha = x_\beta = 1 \otimes \zeta_\beta + I_{\mathbb{D}}$, whence

$$\|x_\alpha\|_{\rho,\tau,t} \le \|\zeta_\beta\|_{\rho,\tau} = \rho^d \tau^{s(\beta)+1} \le \rho^d \tau^{n+3}. \tag{8.14}$$

Raising (8.14) to the power $1/d$, taking the supremum over $\alpha \in W_{n,d}$, and letting $d \to \infty$, we obtain (8.13), as required. ☐

Theorem 8.8. *For each $n \in \mathbb{N}$ and each $r \in (0, +\infty]$, $i_{\mathbb{D}\mathbb{D}} \colon \mathscr{O}_{\mathrm{def}}(\mathbb{D}^n_r) \to \mathscr{O}^{\mathrm{T}}_{\mathrm{def}}(\mathbb{D}^n_r)$ is a topological isomorphism.*

Proof. Let $\varphi_1 \colon \mathscr{O}(\mathbb{C}^\times) \to \mathscr{O}_{\mathrm{def}}(\mathbb{D}^n_r)$ denote the homomorphism given by $\varphi_1(f) = f \otimes 1 + I_{\mathbb{D}}$. By Lemma 8.7, the n-tuple $(x_1, \ldots, x_n) \in \mathscr{O}_{\mathrm{def}}(\mathbb{D}^n_r)^n$ is strictly spectrally r-contractive. Applying Proposition 6.11, we obtain a continuous homomorphism

$$\varphi_2 \colon \mathscr{F}^{\mathrm{T}}(\mathbb{D}^n_r) \to \mathscr{O}_{\mathrm{def}}(\mathbb{D}^n_r), \qquad \zeta_i \mapsto x_i \quad (i = 1, \ldots, n).$$

Since the images of φ_1 and φ_2 commute, the map

$$\varphi \colon \mathscr{O}(\mathbb{C}^\times) \mathbin{\widehat{\otimes}} \mathscr{F}^{\mathrm{T}}(\mathbb{D}^n_r) \to \mathscr{O}_{\mathrm{def}}(\mathbb{D}^n_r), \quad \varphi(f \otimes a) = \varphi_1(f)\varphi_2(a),$$

is an algebra homomorphism. By construction, $\varphi(\zeta_i\zeta_j - z\zeta_j\zeta_i) = 0$ for all $i < j$. Hence φ vanishes on $I^{\mathrm{T}}_{\mathbb{D}}$ and induces a homomorphism

$$\psi \colon \mathscr{O}^{\mathrm{T}}_{\mathrm{def}}(\mathbb{D}^n_r) \to \mathscr{O}_{\mathrm{def}}(\mathbb{D}^n_r).$$

Clearly, ψ takes each x_i to x_i and z to z. Since the canonical images of $\mathscr{O}^{\mathrm{reg}}_{\mathrm{def}}(\mathbb{C}^n)$ are dense in $\mathscr{O}_{\mathrm{def}}(\mathbb{D}^n_r)$ and in $\mathscr{O}^{\mathrm{T}}_{\mathrm{def}}(\mathbb{D}^n_r)$, we conclude that $\psi i_{\mathbb{D}\mathbb{D}}$ and $i_{\mathbb{D}\mathbb{D}}\psi$ are the identity maps on $\mathscr{O}_{\mathrm{def}}(\mathbb{D}^n_r)$ and $\mathscr{O}^{\mathrm{T}}_{\mathrm{def}}(\mathbb{D}^n_r)$, respectively. This completes the proof. ☐

Corollary 8.9. *The Fréchet algebra bundles $\mathsf{E}(\mathbb{D}^n_r)$ and $\mathsf{E}^{\mathrm{T}}(\mathbb{D}^n_r)$ are isomorphic.*

From now on, we identify $\mathscr{O}^{\mathrm{T}}_{\mathrm{def}}(\mathbb{D}^n_r)$ with $\mathscr{O}_{\mathrm{def}}(\mathbb{D}^n_r)$ and $\mathsf{E}^{\mathrm{T}}(\mathbb{D}^n_r)$ with $\mathsf{E}(\mathbb{D}^n_r)$ via $i_{\mathbb{D}\mathbb{D}}$.

8.2. **The nonprojectivity of $\mathscr{O}_{\mathrm{def}}(\mathbb{D}^n_r)$.** In this subsection, we show that $\mathscr{O}_{\mathrm{def}}(\mathbb{D}^n_r)$ is not topologically projective (and, as a consequence, is not topologically free) over $\mathscr{O}(\mathbb{C}^\times)$. Towards this goal, we first give a power series representation of $\mathscr{O}_{\mathrm{def}}(\mathbb{D}^n_r)$, which may be of independent interest.

Define $\omega \colon \mathbb{Z}^n_+ \times \mathbb{Z} \to \mathbb{R}$ by

$$\omega(k,p) = \begin{cases} p & \text{if } p \ge 0; \\ 0 & \text{if } p < 0 \text{ and } p + \sum\limits_{i<j} k_i k_j \ge 0; \\ p + \sum\limits_{i<j} k_i k_j & \text{if } p + \sum\limits_{i<j} k_i k_j < 0. \end{cases}$$

Observe that

$$|\omega(k,p)| = \min\Big\{|\lambda| : \lambda \in \Big[p, p+\sum_{i<j} k_i k_j\Big]\Big\}. \tag{8.15}$$

Lemma 8.10. *For all $k,\ell \in \mathbb{Z}_+^n$, $p,q \in \mathbb{Z}$ we have*

$$|\omega(k+\ell, p+q-\sum_{i>j} k_i\ell_j)| \le |\omega(k,p)| + |\omega(\ell,q)|.$$

Proof. By (8.15), we have

$$\begin{aligned} |\omega(k,p)| + |\omega(\ell,q)| &= \min\Big\{|\lambda|+|\mu| : \lambda \in \Big[p, p+\sum_{i<j} k_i k_j\Big],\ \mu \in \Big[q, q+\sum_{i<j} \ell_i\ell_j\Big]\Big\} \\ &\ge \min\Big\{|\lambda+\mu| : \lambda \in \Big[p, p+\sum_{i<j} k_i k_j\Big],\ \mu \in \Big[q, q+\sum_{i<j} \ell_i\ell_j\Big]\Big\} \\ &= \min\Big\{|\lambda| : \lambda \in \Big[p+q, p+q+\sum_{i<j} k_i k_j + \sum_{i<j} \ell_i\ell_j\Big]\Big\}. \end{aligned} \tag{8.16}$$

On the other hand,

$$\begin{aligned} &|\omega(k+\ell, p+q-\sum_{i>j} k_i\ell_j)| \\ &\quad= \min\Big\{|\lambda| : \lambda \in \Big[p+q-\sum_{i>j} k_i\ell_j, p+q-\sum_{i>j} k_i\ell_j + \sum_{i<j}(k_i+\ell_i)(k_j+\ell_j)\Big]\Big\} \\ &\quad= \min\Big\{|\lambda| : \lambda \in \Big[p+q-\sum_{i>j} k_i\ell_j, p+q+\sum_{i<j} k_ik_j + \sum_{i<j} \ell_i\ell_j + \sum_{i<j} k_i\ell_j\Big]\Big\}. \end{aligned} \tag{8.17}$$

To complete the proof, it remains to observe that the interval over which the minimum is taken in (8.16) is contained in the respective interval in (8.17). $\square$

Given $n \in \mathbb{N}$ and $r \in (0,+\infty]$, let

$$D_{n,r} = \Big\{a = \sum_{k\in\mathbb{Z}_+^n,\, p\in\mathbb{Z}} c_{kp} x^k z^p : \|a\|_{\rho,\tau} = \sum_{k,p} |c_{kp}|\rho^{|k|}\tau^{|\omega(k,p)|} < \infty\ \forall \rho \in (0,r),\ \forall \tau \ge 1\Big\}.$$

Clearly, $D_{n,r}$ is a Fréchet space with respect to the topology determined by the family $\{\|\cdot\|_{\rho,\tau} : \rho \in (0,r),\ \tau \ge 1\}$ of norms. By Lemma 8.4 (i), we may identify $\mathscr{O}_{\mathrm{def}}^{\mathrm{reg}}(\mathbb{C}^n)$ with a dense vector subspace of $D_{n,r}$.

Proposition 8.11. *The multiplication on $\mathscr{O}_{\mathrm{def}}^{\mathrm{reg}}(\mathbb{C}^n)$ uniquely extends to a continuous multiplication on $D_{n,r}$. Moreover, each norm $\|\cdot\|_{\rho,\tau}$ $(\rho \in (0,r),\ \tau \ge 1)$ is submultiplicative on $D_{n,r}$.*

Proof. Since $D_{n,r}$ is the completion of $\mathscr{O}_{\mathrm{def}}^{\mathrm{reg}}(\mathbb{C}^n)$, it suffices to show that every norm $\|\cdot\|_{\rho,\tau}$ $(\rho \in (0,r),\ \tau \ge 1)$ is submultiplicative on $\mathscr{O}_{\mathrm{def}}^{\mathrm{reg}}(\mathbb{C}^n)$. Let $a = x^k z^p$ and $b = x^\ell z^q \in \mathscr{O}_{\mathrm{def}}^{\mathrm{reg}}(\mathbb{C}^n)$. By Lemma 8.10, we have

$$\begin{aligned} \|ab\|_{\rho,\tau} = \|x^{k+\ell} z^{p+q-\sum_{i>j} k_i\ell_j}\|_{\rho,\tau} &= \rho^{|k+\ell|}\tau^{|\omega(k+\ell,p+q-\sum_{i>j} k_i\ell_j)|} \\ &\le \rho^{|k|}\rho^{|\ell|}\tau^{|\omega(k,p)|}\tau^{|\omega(\ell,q)|} = \|a\|_{\rho,\tau}\|b\|_{\rho,\tau}. \end{aligned}$$

Since the norm of every element of $\mathscr{O}_{\mathrm{def}}^{\mathrm{reg}}(\mathbb{C}^n)$ is the sum of the norms of the respective monomials, we conclude that the inequality $\|ab\|_{\rho,\tau} \le \|a\|_{\rho,\tau}\|b\|_{\rho,\tau}$ holds for all $a,b \in \mathscr{O}_{\mathrm{def}}^{\mathrm{reg}}(\mathbb{C}^n)$. $\square$

Theorem 8.12. *There exists a topological algebra isomorphism $\mathscr{O}_{\mathrm{def}}(\mathbb{D}_r^n) \to D_{n,r}$ uniquely determined by $z \mapsto z$, $x_i \mapsto x_i$ $(i = 1, \dots, n)$.*

Proof. Since $D_{n,r}$ is an Arens-Michael algebra, and since $z \in D_{n,r}$ is invertible, there exists a unique continuous homomorphism $\varphi_1 \colon \mathscr{O}(\mathbb{C}^\times) \to D_{n,r}$ such that $\varphi_1(w) = z$ (where w is the coordinate on $\mathbb{C}$). Observe also that for each $i = 1, \dots, n$ we have a continuous homomorphism from $\mathscr{O}(\mathbb{D}_r)$ to $D_{n,r}$ uniquely determined by $w \mapsto x_i$ (cf. (3.1)). By the universal property of $\mathscr{F}(\mathbb{D}_r^n)$ (see (6.1)), there exists a unique continuous homomorphism $\varphi_2 \colon \mathscr{F}(\mathbb{D}_r^n) \to D_{n,r}$ such that $\varphi_2(\zeta_i) = x_i$ $(i = 1, \dots, n)$. Since the images of φ_1 and φ_2 commute, the map

$$\tilde{\varphi} \colon \mathscr{O}(\mathbb{C}^\times) \mathbin{\widehat{\otimes}} \mathscr{F}(\mathbb{D}_r^n) \to D_{n,r}, \quad f \otimes a \mapsto \varphi_1(f)\varphi_2(a),$$

is an algebra homomorphism. By construction, $\tilde{\varphi}(I_{\mathbb{D}}) = 0$. Hence $\tilde{\varphi}$ induces a continuous homomorphism

$$\varphi \colon \mathscr{O}_{\mathrm{def}}(\mathbb{D}_r^n) \to D_{n,r}, \quad z \mapsto z, \quad x_i \mapsto x_i \quad (i = 1, \dots, n).$$

We claim that φ is a topological isomorphism. To see this, observe first that for each $k \in \mathbb{Z}_+^n$ and each $p \in \mathbb{Z}$ we have $\omega(k,p) - p \in [0, \sum_{i<j} k_i k_j]$. By Lemma 8.5, there exists $\alpha(k,p) \in W_n$ such that

$$\mathrm{p}(\alpha(k,p)) = k, \quad \mathrm{s}(\alpha(k,p)) \le n+2, \quad x^k = x_{\alpha(k,p)} z^{\omega(k,p)-p}. \tag{8.18}$$

We now define

$$\tilde{\psi} \colon D_{n,r} \to \mathscr{O}(\mathbb{C}^\times) \mathbin{\widehat{\otimes}} \mathscr{F}(\mathbb{D}_r^n), \quad \sum_{k,p} c_{kp} x^k z^p \mapsto \sum_{k,p} c_{kp} z^{\omega(k,p)} \otimes \zeta_{\alpha(k,p)}.$$

To see that $\tilde{\psi}$ is a continuous linear map from $D_{n,r}$ to $\mathscr{O}(\mathbb{C}^\times) \mathbin{\widehat{\otimes}} \mathscr{F}(\mathbb{D}_r^n)$, take $t, \tau \ge 1$, $\rho \in (0, r)$, and let $\|\cdot\|_{t;\rho,\tau}$ denote the projective tensor norm $\|\cdot\|_t \otimes_\pi \|\cdot\|_{\rho,\tau}$ on $\mathscr{O}(\mathbb{C}^\times) \mathbin{\widehat{\otimes}} \mathscr{F}(\mathbb{D}_r^n)$ (cf. Lemma 8.7). Using the first two formulas in (8.18), we see that

$$\sum_{k,p} |c_{kp}| \| z^{\omega(k,p)} \otimes \zeta_{\alpha(k,p)} \|_{t;\rho,\tau} \le \sum_{k,p} |c_{kp}| t^{|\omega(k,p)|} \rho^{|k|} \tau^{n+3} = \tau^{n+3} \Big\| \sum_{k,p} c_{kp} x^k z^p \Big\|_{\rho,t}.$$

This implies that $\tilde{\psi}$ indeed takes $D_{n,r}$ to $\mathscr{O}(\mathbb{C}^\times) \mathbin{\widehat{\otimes}} \mathscr{F}(\mathbb{D}_r^n)$ and is continuous. Let now $\psi \colon D_{n,r} \to \mathscr{O}_{\mathrm{def}}(\mathbb{D}_r^n)$ be the composition of $\tilde{\psi}$ and the quotient map of $\mathscr{O}(\mathbb{C}^\times) \mathbin{\widehat{\otimes}} \mathscr{F}(\mathbb{D}_r^n)$ onto $\mathscr{O}_{\mathrm{def}}(\mathbb{D}_r^n)$. By the third equality in (8.18), we have

$$\psi(x^k z^p) = x_{\alpha(k,p)} z^{\omega(k,p)} = x^k z^p \qquad (k \in \mathbb{Z}_+^n,\ p \in \mathbb{Z}).$$

Since $\varphi(x^k z^p) = x^k z^p$ as well, and since the monomials $x^k z^p$ span dense vector subspaces of $D_{n,r}$ and $\mathscr{O}_{\mathrm{def}}(\mathbb{D}_r^n)$, we conclude that $\varphi\psi$ and $\psi\varphi$ are the identity maps. This completes the proof. $\square$

Recall some definitions and facts from [59] (see also [60]). Let A be a Fréchet algebra. By a *left Fréchet A-module* we mean a left A-module X together with a Fréchet space topology such that the action $A \times X \to X$ is continuous. Morphisms of Fréchet A-modules are assumed to be continuous. A morphism $\sigma \colon X \to Y$ of left Fréchet A-modules is an *admissible epimorphism* if there exists a continuous linear map $\varkappa \colon Y \to X$ such that $\sigma\varkappa = 1_Y$. A left Fréchet A-module P is (topologically) *projective* if for each admissible epimorphism $X \to Y$ of left Fréchet A-modules the induced map $\mathrm{Hom}_A(P, X) \to \mathrm{Hom}_A(P, Y)$ is onto. If E is a Fréchet space, then the projective tensor product $A \mathbin{\widehat{\otimes}} E$ is a left Fréchet A-module in a natural way. A left Fréchet A-module F is (topologically) *free* if $F \cong A \mathbin{\widehat{\otimes}} E$ for some

Fréchet space E. Since $\mathrm{Hom}_A(A \mathbin{\widehat{\otimes}} E, -) \cong \mathrm{Hom}_{\mathbb{C}}(E, -)$, it follows that each free Fréchet A-module is projective. Given a left Fréchet A-module P, let $\mu_P \colon A \mathbin{\widehat{\otimes}} P \to P$ denote the A-module morphism uniquely determined by $a \otimes x \mapsto ax$. By [59, Chap. III, Theorem 1.27], P is projective if and only if there exists an A-module morphism $\nu \colon P \to A \mathbin{\widehat{\otimes}} P$ such that $\mu_P \nu = 1_P$.

Theorem 8.13. *For each $n \geq 2$ and each $r \in (0, +\infty]$, the Fréchet $\mathscr{O}(\mathbb{C}^\times)$-module $\mathscr{O}_{\mathrm{def}}(\mathbb{D}^n_r)$ is not projective (and hence is not free).*

Proof. We identify $\mathscr{O}_{\mathrm{def}}(\mathbb{D}^n_r)$ with $D_{n,r}$ via the isomorphism constructed in Theorem 8.12. Assume, towards a contradiction, that $\mathscr{O}_{\mathrm{def}}(\mathbb{D}^n_r)$ is projective over $\mathscr{O}(\mathbb{C}^\times)$. Then there exists a Fréchet $\mathscr{O}(\mathbb{C}^\times)$-module morphism $\nu \colon \mathscr{O}_{\mathrm{def}}(\mathbb{D}^n_r) \to \mathscr{O}(\mathbb{C}^\times) \mathbin{\widehat{\otimes}} \mathscr{O}_{\mathrm{def}}(\mathbb{D}^n_r)$ such that $\mu\nu = 1$ (where $\mu = \mu_{\mathscr{O}_{\mathrm{def}}(\mathbb{D}^n_r)}$). Recall the standard fact that the projective tensor product of two Köthe sequence spaces is again a Köthe sequence space (cf. [81, 41.7]). Hence we have a topological isomorphism

$$
\begin{aligned}
&\mathscr{O}(\mathbb{C}^\times) \mathbin{\widehat{\otimes}} \mathscr{O}_{\mathrm{def}}(\mathbb{D}^n_r) \\
&\cong \Big\{ v = \sum_{\substack{k \in \mathbb{Z}^n_+ \\ r,p \in \mathbb{Z}}} c_{rkp} z^r \otimes x^k z^p : \|v\|_{t,\rho,\tau} = \sum_{r,k,p} |c_{rkp}| t^{|r|} \rho^{|k|} \tau^{|\omega(k,p)|} < \infty \ \forall \rho \in (0,r),\ \forall t, \tau \geq 1 \Big\}.
\end{aligned}
$$

For each $m \in \mathbb{N}$, let

$$\nu(x_1^{2m} x_2^m) = \sum_{r,k,p} c^{(m)}_{rkp} z^r \otimes x^k z^p.$$

Since $\mu\nu = 1$, we see that $c^{(m)}_{rkp} = 0$ unless $p + r = 0$. Hence

$$\nu(x_1^{2m} x_2^m) = \sum_{k,p} c^{(m)}_{kp} z^{-p} \otimes x^k z^p, \tag{8.19}$$

where $c^{(m)}_{kp} = c^{(m)}_{-pkp}$. This implies that

$$x_1^{2m} x_2^m = \mu\Big(\sum_{k,p} c^{(m)}_{kp} z^{-p} \otimes x^k z^p \Big) = \sum_{k,p} c^{(m)}_{kp} x^k.$$

Letting $\bar{m} = (2m, m, 0, \ldots, 0) \in \mathbb{Z}^n_+$, we conclude that

$$\sum_p c^{(m)}_{kp} = \begin{cases} 1 & \text{if } k = \bar{m}; \\ 0 & \text{if } k \neq \bar{m}. \end{cases} \tag{8.20}$$

Fix $\rho \in (0, r)$ and choose $\rho_1 \in (0, r)$, $\tau_1 \geq 1$, and $C > 0$ such that

$$\|\nu(u)\|_{2,\rho,1} \leq C \|u\|_{\rho_1,\tau_1} \qquad (u \in \mathscr{O}_{\mathrm{def}}(\mathbb{D}^n_r)). \tag{8.21}$$

Letting $u_{sm} = z^{-s} x_1^{2m} x_2^m$, where $0 \leq s \leq 2m^2$, we obtain from (8.21)

$$\|\nu(u_{sm})\|_{2,\rho,1} \leq C \|u_{sm}\|_{\rho_1,\tau_1} = C \rho_1^{3m}, \tag{8.22}$$

because $\omega(\bar{m}, -s) = 0$. On the other hand, (8.19) implies that

$$
\begin{aligned}
\|\nu(u_{sm})\|_{2,\rho,1} = \|z^{-s} \nu(x_1^{2m} x_2^m)\|_{2,\rho,1} &= \Big\| \sum_{k,p} c^{(m)}_{kp} z^{-s-p} \otimes x^k z^p \Big\|_{2,\rho,1} \\
&= \sum_{k,p} |c^{(m)}_{kp}| 2^{|s+p|} \rho^{|k|} \geq \sum_p |c^{(m)}_{\bar{m}p}| 2^{|s+p|} \rho^{3m}.
\end{aligned} \tag{8.23}
$$

Combining (8.22) and (8.23), we see that

$$\sum_p |c_{\bar{m}p}^{(m)}| 2^{|s+p|} \rho^{3m} \le C\rho_1^{3m} \qquad (m \in \mathbb{N},\ 0 \le s \le 2m^2). \tag{8.24}$$

Since $\sum_{p\in\mathbb{Z}} |c_{\bar{m}p}^{(m)}| \ge 1$ by (8.20), we have

$$\text{either} \sum_{p\ge -m^2} |c_{\bar{m}p}^{(m)}| \ge 1/2 \tag{8.25}$$

$$\text{or} \sum_{p< -m^2} |c_{\bar{m}p}^{(m)}| \ge 1/2. \tag{8.26}$$

If (8.25) holds, then, letting $s = 2m^2$ in (8.24), we obtain

$$C\rho_1^{3m} \ge \sum_{p\ge -m^2} |c_{\bar{m}p}^{(m)}| 2^{|2m^2+p|} \rho^{3m} \ge \frac{1}{2}\, 2^{m^2} \rho^{3m}.$$

On the other hand, if (8.26) holds, then, letting $s = 0$ in (8.24), we see that

$$C\rho_1^{3m} \ge \sum_{p< -m^2} |c_{\bar{m}p}^{(m)}| 2^{|p|} \rho^{3m} \ge \frac{1}{2}\, 2^{m^2} \rho^{3m}.$$

Thus for each $m \in \mathbb{N}$ we have $2^{m^2-1} \le C(\rho_1/\rho)^{3m}$, which is impossible. The resulting contradiction completes the proof. $\square$

Remark 8.14. In [103, Example 3.2], the authors construct the algebra $\mathscr{O}_{\mathrm{def}}(\mathbb{C}^n)$ and claim (essentially without proof) that it is free over $\mathscr{O}(\mathbb{C}^\times)$. Theorem 8.13 shows that this is not the case.

Remark 8.15. We conjecture that a result similar to Theorem 8.13 holds for $\mathscr{O}_{\mathrm{def}}(\mathbb{B}_r^n)$ as well.

8.3. **The continuity of $\mathsf{E}(\mathbb{D}_r^n)$ and $\mathsf{E}(\mathbb{B}_r^n)$.** In this subsection, our goal is to show that the Fréchet algebra bundles $\mathsf{E}(\mathbb{D}_r^n)$ and $\mathsf{E}(\mathbb{B}_r^n)$ are continuous. This will be deduced from the following general result.

Theorem 8.16. *Let X be a reduced Stein space, F be a Fréchet algebra, $I \subset \mathscr{O}(X, F)$ be a closed two-sided ideal, and $A = \mathscr{O}(X, F)/I$. For every $x \in X$, we identify A_x with F/I_x by Lemma 8.2. Suppose that there exist a dense subalgebra $A_0 \subset A$ and a directed defining family $\mathscr{N}_F = \{\|\cdot\|_\lambda : \lambda \in \Lambda\}$ of seminorms on F such that for each $a \in A_0$ and each $\lambda \in \Lambda$ the function $X \to \mathbb{R}$, $x \mapsto \|a_x\|_{\lambda,x}$, is continuous (where $\|\cdot\|_{\lambda,x}$ is the quotient seminorm of $\|\cdot\|_\lambda$ on F/I_x). Then the bundle $\mathsf{E}(A)$ is continuous.*

To prove Theorem 8.16, we need two lemmas.

Lemma 8.17. *Under the conditions of Lemma 8.1, suppose that $\|\cdot\|$ is a continuous seminorm on $\mathfrak{A}$. Let $\|\cdot\|_I$ be the quotient seminorm of $\|\cdot\|$ on $\mathfrak{A}/I$, and let $\|\cdot\|_{I,J}$ be the quotient seminorm of $\|\cdot\|_I$ on $(\mathfrak{A}/I)/J_0$. Then the isomorphisms (8.2) are isometric with respect to $\|\cdot\|_{\overline{I+J}}$, $\|\cdot\|_{I,J}$, and $\|\cdot\|_{J,I}$, respectively.*

Proof. Elementary. $\square$

Lemma 8.18. *Let $K \subset X$ be a holomorphically convex compact set. For each $f \in \mathscr{O}(X)$, we let $\|f\|_K = \sup_{x\in K} |f(x)|$. Given $\lambda \in \Lambda$, let $\|\cdot\|^\pi_{K,\lambda}$ denote the projective tensor seminorm $\|\cdot\|_K \otimes_\pi \|\cdot\|_\lambda$ on $\mathscr{O}(X,F)$, and let $\|\cdot\|_{K,\lambda}$ denote the quotient seminorm of $\|\cdot\|^\pi_{K,\lambda}$ on A. Finally, given $x \in X$, let $\|\cdot\|_{K,\lambda,x}$ denote the quotient seminorm of $\|\cdot\|_{K,\lambda}$ on A_x. Then*

$$\|\cdot\|_{K,\lambda,x} = \begin{cases} \|\cdot\|_{\lambda,x} & \text{if } x \in K; \\ 0 & \text{if } x \notin K. \end{cases} \tag{8.27}$$

Proof. As in Lemma 8.2, we identify F with $\mathscr{O}(X,F)/\operatorname{Ker}\varepsilon^F_x$. Let $\|\cdot\|^{(x)}_{K,\lambda}$ denote the quotient seminorm of $\|\cdot\|^\pi_{K,\lambda}$ on F. By applying Lemma 8.17 to $\mathfrak{A} = \mathscr{O}(X,F)$ and $J = \overline{\mathfrak{m}_x\mathfrak{A}}$ (see (8.3)), we conclude that $\|\cdot\|_{K,\lambda,x}$ equals the quotient seminorm of $\|\cdot\|^{(x)}_{K,\lambda}$ on $F/I_x = A_x$. To complete the proof, it remains to compare the seminorms $\|\cdot\|^{(x)}_{K,\lambda}$ and $\|\cdot\|_\lambda$ on F, i.e., to show that

$$\|\cdot\|^{(x)}_{K,\lambda} = \begin{cases} \|\cdot\|_{\lambda} & \text{if } x \in K, \\ 0 & \text{if } x \notin K. \end{cases} \tag{8.28}$$

Observe that $\varepsilon^F_x = \varepsilon_x \otimes 1_F$, where $\varepsilon_x = \varepsilon^{\mathbb{C}}_x \colon \mathscr{O}(X) \to \mathbb{C}$ is the evaluation map. If $x \in K$, then for every $f \in \mathscr{O}(X)$ we clearly have $|\varepsilon_x(f)| \le \|f\|_K$. This implies that $\|\varepsilon^F_x(u)\|_\lambda \le \|u\|^\pi_{K,\lambda}$ for each $u \in \mathscr{O}(X,F)$. Hence $\|\cdot\|_\lambda \le \|\cdot\|^{(x)}_{K,\lambda}$. On the other hand, for each $v \in F$ we have $v = \varepsilon^F_x(1\otimes v)$, and $\|1\otimes v\|^\pi_{K,\lambda} = \|v\|_\lambda$. Therefore $\|\cdot\|_\lambda = \|\cdot\|^{(x)}_{K,\lambda}$ whenever $x \in K$.

Now assume that $x \notin K$. Since K is holomorphically convex, there exists $f \in \mathscr{O}(X)$ such that $f(x) = 1$ and $\|f\|_K < 1$. For each $v \in F$ and each $n \in \mathbb{N}$ we have $v = f^n(x)v = \varepsilon^F_x(f^n \otimes v)$, whence

$$\|v\|^{(x)}_{K,\lambda} \le \|f^n \otimes v\|^\pi_{K,\lambda} \le \|f\|^n_K \|v\|_\lambda \to 0 \quad (n \to \infty).$$

Thus $\|\cdot\|^{(x)}_{K,\lambda} = 0$ whenever $x \notin K$. This implies (8.28) and completes the proof. ☐

Proof of Theorem 8.16. By construction of $\mathsf{E}(A)$ and by Remark A.26, the locally convex uniform vector structure on $\mathsf{E}(A)$ is given by the family

$$\mathscr{N}_A = \{\|\cdot\|_{K,\lambda} : K \in \mathrm{HCC}(X),\ \lambda \in \Lambda\},$$

where $\mathrm{HCC}(X)$ denotes the collection of all holomorphically convex compact subsets of X, and $\|\cdot\|_{K,\lambda}$ is the seminorm on $\mathsf{E}(A)$ whose restriction to each fiber A_x is $\|\cdot\|_{K,\lambda,x}$. Let $\Gamma = \{\tilde a : a \in A_0\} \subset \Gamma(X,E)$. Since A_0 is dense in A, it follows that the set $\{s_x : s \in \Gamma\}$ is dense in A_x for each $x \in X$. Unfortunately, we cannot directly apply Proposition A.33 to Γ and $\|\cdot\|_{K,\lambda}$ because of (8.27), which implies that the function $x \mapsto \|s_x\|_{K,\lambda} = \|s_x\|_{K,\lambda,x}$ is continuous on K, but, in general, not on the whole of X. Thus we have to modify $\mathscr{N}_A$ as follows. Given $K \in \mathrm{HCC}(X)$, choose a continuous, compactly supported function $h_K \colon X \to [0,1]$ such that $h_K(x) = 1$ for all $x \in K$, and let K' denote the holomorphically convex hull of $\operatorname{supp} h_K$. Define a new seminorm $\|\cdot\|'_{K,\lambda}$ on $\mathsf{E}(A)$ by

$$\|u\|'_{K,\lambda} = h_K(p(u))\|u\|_{K',\lambda} \qquad (u \in \mathsf{E}(A)),$$

and let $\mathscr{N}'_A = \{\|\cdot\|'_{K,\lambda} : K \in \mathrm{HCC}(X),\ \lambda \in \Lambda\}$. Clearly, $\|\cdot\|'_{K,\lambda}$ is upper semicontinuous (being the product of two nonnegative, upper semicontinuous functions). Taking into

account (8.27), we see that

$$\|\cdot\|_{K,\lambda} \le \|\cdot\|'_{K,\lambda} \le \|\cdot\|_{K',\lambda}.$$

This implies that $\mathscr{N}'_A \sim \mathscr{N}_A$, whence $\mathscr{U}_{\mathscr{N}_A} = \mathscr{U}_{\mathscr{N}'_A}$. Moreover, $\mathscr{N}'_A$ is admissible by Lemma A.16 (ii). Let now $a \in A_0$ and $x \in X$. By (8.27) and by the choice of h_K, we see that

$$\|\tilde{a}_x\|'_{K,\lambda} = \|a_x\|'_{K,\lambda} = h_K(x)\|a_x\|_{K',\lambda} = h_K(x)\|a_x\|_{\lambda,x},$$

which is a continuous function on X by assumption. Now the result follows from Proposition A.33 applied to Γ and $\mathscr{N}'_A$. $\square$

Corollary 8.19. *The bundles* $\mathsf{E}(\mathbb{D}^n_r)$ *and* $\mathsf{E}(\mathbb{B}^n_r)$ *are continuous.*

Proof. For convenience, let us denote the norm $\|\cdot\|_{\mathbb{D},\rho}$ on $\mathscr{O}_q(\mathbb{D}^n_r)$ by $\|\cdot\|_{\mathbb{D},q,\rho}$. Similarly, we write $\|\cdot\|_{\mathbb{B},q,\rho}$ for the norm $\|\cdot\|_{\mathbb{B},\rho}$ on $\mathscr{O}_q(\mathbb{B}^n_r)$. Let $F = \mathscr{F}^{\mathrm{T}}(\mathbb{D}^n_r)$, and let $\{\|\cdot\|_\rho : \rho \in (0,r)\}$ be the standard defining family of seminorms on F, where $\|\cdot\|_\rho$ is given by (6.3). Let also $I = I^{\mathrm{T}}_{\mathbb{D}}$ and $A = \mathscr{O}(\mathbb{C}^\times, F)/I = \mathscr{O}^{\mathrm{T}}_{\mathrm{def}}(\mathbb{D}^n_r) \cong \mathscr{O}_{\mathrm{def}}(\mathbb{D}^n_r)$ (see Theorem 8.8). As in Lemma 8.2, given $q \in \mathbb{C}^\times$, let $I_q = \overline{\varepsilon^F_q(I)} \subset F$. By Theorem 6.14, we can identify F/I_q with $\mathscr{O}_q(\mathbb{D}^n_r)$. Moreover, the quotient seminorm $\|\cdot\|_{\rho,q}$ of $\|\cdot\|_\rho$ on F/I_q becomes $\|\cdot\|_{\mathbb{D},q,\rho}$ under this identification.

Let now $A_0 = \mathscr{O}^{\mathrm{reg}}_{\mathrm{def}}(\mathbb{C}^n)$. By Lemma 8.4, A_0 is a dense subalgebra of A. Given $a = \sum_{k,p} c_{kp} x^k z^p \in A_0$, we have

$$\|a_q\|_{\rho,q} = \|a_q\|_{\mathbb{D},q,\rho} = \sum_k \Bigl|\sum_p c_{kp} q^p\Bigr| w_q(k)\rho^{|k|}.$$

This implies that the function $q \mapsto \|a_q\|_{\rho,q}$ is continuous on $\mathbb{C}^\times$. By applying Theorem 8.16, we conclude that $\mathsf{E}(\mathbb{D}^n_r)$ is continuous.

A similar argument applies to $\mathsf{E}(\mathbb{B}^n_r)$. Specifically, we have to replace $\mathscr{F}^{\mathrm{T}}(\mathbb{D}^n_r)$ by $\mathscr{F}(\mathbb{B}^n_r)$, $\|\cdot\|_\rho$ by $\|\cdot\|^\circ_\rho$, $I^{\mathrm{T}}_{\mathbb{D}}$ by $I_{\mathbb{B}}$, and to apply Theorem 7.10 instead of Theorem 6.14. The continuity of $q \mapsto \|a_q\|^\circ_{\rho,q}$ now follows from

$$\|a_q\|^\circ_{\rho,q} = \|a_q\|_{\mathbb{B},q,\rho} = \sum_k \Bigl|\sum_p c_{kp} q^p\Bigr| \left(\frac{[k]_{|q|^2}!}{\bigl[|k|\bigr]_{|q|^2}!}\right)^{1/2} u_q(k)\rho^{|k|}. \qquad \square$$

8.4. **Relations to Rieffel's quantization.** The strict (C^*-algebraic) version of deformation quantization was introduced by M. Rieffel [136] (see also [137–141]). In this subsection, we show that the algebras $\mathscr{O}_q(\mathbb{D}^n_r)$ and $\mathscr{O}_q(\mathbb{B}^n_r)$ fit into Rieffel's framework adapted to the Fréchet algebra setting.

Towards this goal, it will be convenient to modify the bundles $\mathsf{E}(\mathbb{D}^n_r)$ and $\mathsf{E}(\mathbb{B}^n_r)$ by replacing the "deformation parameter" $q \in \mathbb{C}^\times$ by $h \in \mathbb{C}$, where $q = \exp(ih)$. Specifically, let $h \in \mathscr{O}(\mathbb{C})$ denote the complex coordinate, and let $\tilde{I}_{\mathbb{D}}$, $\tilde{I}^{\mathrm{T}}_{\mathbb{D}}$, and $\tilde{I}_{\mathbb{B}}$ denote the closed two-sided ideals of $\mathscr{O}(\mathbb{C}, \mathscr{F}(\mathbb{D}^n_r))$, $\mathscr{O}(\mathbb{C}, \mathscr{F}^{\mathrm{T}}(\mathbb{D}^n_r))$, and $\mathscr{O}(\mathbb{C}, \mathscr{F}(\mathbb{B}^n_r))$, respectively, generated by the elements $\zeta_j\zeta_k - e^{ih}\zeta_k\zeta_j$ $(j < k)$. By analogy with (8.1), consider the Fréchet $\mathscr{O}(\mathbb{C})$-algebras

$$\begin{aligned}
\tilde{\mathscr{O}}_{\mathrm{def}}(\mathbb{D}^n_r) &= \mathscr{O}(\mathbb{C}, \mathscr{F}(\mathbb{D}^n_r))/\tilde{I}_{\mathbb{D}},\\
\tilde{\mathscr{O}}^{\mathrm{T}}_{\mathrm{def}}(\mathbb{D}^n_r) &= \mathscr{O}(\mathbb{C}, \mathscr{F}^{\mathrm{T}}(\mathbb{D}^n_r))/\tilde{I}^{\mathrm{T}}_{\mathbb{D}},\\
\tilde{\mathscr{O}}_{\mathrm{def}}(\mathbb{B}^n_r) &= \mathscr{O}(\mathbb{C}, \mathscr{F}(\mathbb{B}^n_r))/\tilde{I}_{\mathbb{B}}.
\end{aligned}$$

We will use the following simplified notation for the respective Fréchet algebra bundles:

$$\tilde{\mathsf{E}}(\mathbb{D}^n_r) = \mathsf{E}(\tilde{\mathscr{O}}_{\mathrm{def}}(\mathbb{D}^n_r)), \quad \tilde{\mathsf{E}}^{\mathrm{T}}(\mathbb{D}^n_r) = \mathsf{E}(\tilde{\mathscr{O}}^{\mathrm{T}}_{\mathrm{def}}(\mathbb{D}^n_r)), \quad \tilde{\mathsf{E}}(\mathbb{B}^n_r) = \mathsf{E}(\tilde{\mathscr{O}}_{\mathrm{def}}(\mathbb{B}^n_r)).$$

Exactly as in Subsection 8.1, we see that the fibers of $\tilde{\mathsf{E}}(\mathbb{D}^n_r)$ and $\tilde{\mathsf{E}}(\mathbb{B}^n_r)$ over $h \in \mathbb{C}$ are isomorphic to $\mathscr{O}_{\exp(ih)}(\mathbb{D}^n_r)$ and $\mathscr{O}_{\exp(ih)}(\mathbb{B}^n_r)$, respectively, that $\tilde{\mathsf{E}}(\mathbb{D}^n_r)$ and $\tilde{\mathsf{E}}^{\mathrm{T}}(\mathbb{D}^n_r)$ are isomorphic, and that $\tilde{\mathsf{E}}(\mathbb{D}^n_r)$ and $\tilde{\mathsf{E}}(\mathbb{B}^n_r)$ are continuous.

Remark 8.20. Alternatively, we can define the bundles $\tilde{\mathsf{E}}(\mathbb{D}^n_r)$ and $\tilde{\mathsf{E}}(\mathbb{B}^n_r)$ to be the pullbacks of $\mathsf{E}(\mathbb{D}^n_r)$ and $\mathsf{E}(\mathbb{B}^n_r)$ under the exponential map $e\colon \mathbb{C} \to \mathbb{C}^\times,\ h \mapsto \exp(ih)$. Specifically, suppose that X and Y are topological spaces, $f\colon X \to Y$ is a continuous map, and $(E, p, \mathscr{U})$ is a locally convex bundle over Y. The pullback $f^*E = E \times_Y X$ is a prebundle of topological vector spaces over X in a canonical way; the projection $\tilde{p}\colon f^*E \to X$ is given by $\tilde{p}(v, x) = x$ (cf. [65, 2.5]). Define $\tilde{f}\colon f^*E \to E$ by $\tilde{f}(v, x) = v$, and let $\tilde{f}^{-1}(\mathscr{U})$ denote the locally convex uniform vector structure on f^*E with base $\{\tilde{f}^{-1}(U) : U \in \mathscr{U}\}$. It is easy to show that $(f^*E, \tilde{p}, \tilde{f}^{-1}(\mathscr{U}))$ is a locally convex bundle. Moreover, if $\mathscr{N} = \{\|\cdot\|_\lambda : \lambda \in \Lambda\}$ is an admissible family of seminorms for E, then the collection $\tilde{\mathscr{N}} = \{\|\cdot\|^f_\lambda : \lambda \in \Lambda\}$ is an admissible family of seminorms for f^*E, where $\|\cdot\|^f_\lambda$ is given by $\|(v, x)\|^f_\lambda = \|v\|_\lambda$. Clearly, this implies that if E is continuous, then so is f^*E. Finally, it can be shown that $\tilde{\mathsf{E}}(\mathbb{D}^n_r) \cong e^*\mathsf{E}(\mathbb{D}^n_r)$ and $\tilde{\mathsf{E}}(\mathbb{B}^n_r) \cong e^*\mathsf{E}(\mathbb{B}^n_r)$. We will not use these results below, so we omit the details.

Recall (see, e.g., [82]) that a *Poisson algebra* is a commutative algebra $\mathscr{A}$ together with a bilinear operation $\{\cdot,\cdot\}\colon \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ making $\mathscr{A}$ into a Lie algebra and such that for each $a \in \mathscr{A}$ the map $\{a, \cdot\}$ is a derivation of the associative algebra $\mathscr{A}$.

The following definition is a straightforward modification of Rieffel's quantization to the Fréchet algebra case.

Definition 8.21. Let $\mathscr{A}$ be a Poisson algebra, and let X be an open connected subset of $\mathbb{C}$ containing 0. A *strict Fréchet deformation quantization* of $\mathscr{A}$ is the following data:

(DQ1) A continuous Fréchet algebra bundle $(A, p, \mathscr{U})$ over X;
(DQ2) A family of dense subalgebras $\{\mathscr{A}_h \subset A_h : h \in X\}$;
(DQ3) A family of vector space isomorphisms

$$i_h\colon \mathscr{A} \to \mathscr{A}_h, \quad a \mapsto a_h \qquad (h \in X)$$

such that i_0 is an algebra isomorphism.

Moreover, we require that for each $a, b \in \mathscr{A}$

$$\frac{a_h b_h - b_h a_h}{h} - i\{a, b\}_h \to 0_0 \quad (h \to 0). \tag{8.29}$$

Remark 8.22. If $\{\|\cdot\|_\lambda : \lambda \in \Lambda\}$ is an admissible family of seminorms on A, then (8.29) is equivalent to

$$\left\| \frac{a_h b_h - b_h a_h}{h} - i\{a, b\}_h \right\|_\lambda \to 0 \quad (h \to 0),$$

for all $\lambda \in \Lambda$. This is immediate from the fact that the family

$$\bigl\{\mathsf{T}(V, 0, \lambda, \varepsilon) : V \subset X \text{ is an open neighborhood of } 0,\ \lambda \in \Lambda,\ \varepsilon > 0\bigr\}$$

is a local base at 0_0 (see Lemma A.12 (ii)).

Theorem 8.23. *Consider the Poisson algebra $\mathscr{A} = \mathbb{C}[x_1, \ldots, x_n]$ with bracket*

$$\{f, g\} = \sum_{i<j} x_i x_j \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_i} \right) \qquad (f, g \in \mathscr{A}).$$

Let $(A, p, \mathscr{U})$ be a continuous Fréchet algebra bundle over an open connected subset $X \subset \mathbb{C}$ containing 0. Assume that for each $h \in X$ the fiber A_h contains $\mathscr{A}_h = \mathscr{O}^{\mathrm{reg}}_{\exp(ih)}(\mathbb{C}^n)$ as a dense subalgebra, and that for each $j = 1, \ldots, n$ the constant section x_j of A is continuous (for example, we can let $A = \tilde{\mathsf{E}}(\mathbb{D}^n_r)$ or $A = \tilde{\mathsf{E}}(\mathbb{B}^n_r)$). Let $i_h \colon \mathscr{A} \to \mathscr{A}_h$ be the vector space isomorphism given by $x^k \mapsto x^k$ ($k \in \mathbb{Z}^n_+$). Then $\big(A,\ \{\mathscr{A}_h,\ i_h\}_{h \in X}\big)$ is a strict Fréchet deformation quantization of $\mathscr{A}$.

Proof. The only thing that needs to be proved is the compatibility relation (8.29). Let

$$\sigma \colon \mathbb{Z}^n_+ \times \mathbb{Z}^n_+ \to \mathbb{Z}, \quad \sigma(k, \ell) = \sum_{i<j} k_i \ell_j. \tag{8.30}$$

An easy computation shows that for each $f = \sum_k a_k x^k$ and $g = \sum_k b_k x^k \in \mathscr{A}$ we have

$$f_h g_h = \sum_m \left(\sum_{k+\ell=m} a_k b_\ell e^{-ih\sigma(\ell,k)} \right) (x^m)_h$$

and

$$\{f, g\} = \sum_m \left(\sum_{k+\ell=m} a_k b_\ell \big(\sigma(k,\ell) - \sigma(\ell,k)\big) \right) x^m.$$

Hence

$$\frac{f_h g_h - g_h f_h}{h} - i\{f, g\}_h = \sum_m \left(\sum_{k+\ell=m} a_k b_\ell \varphi_{k\ell}(h) \right) (x^m)_h,$$

where

$$\varphi_{k\ell}(h) = \frac{e^{-ih\sigma(\ell,k)} - e^{-ih\sigma(k,\ell)}}{h} - i\big(\sigma(k,\ell) - \sigma(\ell,k)\big).$$

For each $k, \ell \in \mathbb{Z}^n_+$ we clearly have $\varphi_{k\ell}(h) \to 0$ as $h \to 0$.

Suppose now that $\{\|\cdot\|_\lambda : \lambda \in \Lambda\}$ is an admissible family of continuous seminorms on A. For each $m \in \mathbb{Z}^n_+$ and each $\lambda \in \Lambda$, the function $h \mapsto \|(x^m)_h\|_\lambda$ is continuous and hence is locally bounded. Therefore

$$\left\| \frac{f_h g_h - g_h f_h}{h} - i\{f, g\}_h \right\|_\lambda \le \sum_m \sum_{k+\ell=m} |a_k b_\ell \varphi_{k\ell}(h)| \|(x^m)_h\|_\lambda \to 0 \quad (h \to 0).$$

By Remark 8.22, this implies (8.29) and completes the proof. ☐

8.5. **Relations to formal deformations.** In this subsection, we associate to $\mathscr{O}_{\mathrm{def}}(\mathbb{D}^n_r)$ and $\mathscr{O}_{\mathrm{def}}(\mathbb{B}^n_r)$ two Fréchet $\mathbb{C}[[h]]$-algebras that are formal deformations (in the sense of [29]) of $\mathscr{O}(\mathbb{D}^n_r)$ and $\mathscr{O}(\mathbb{B}^n_r)$, respectively.

Let A be a Fréchet algebra. Following [29], we say that a Fréchet $\mathbb{C}[[h]]$-algebra $\tilde{A}$ is a *formal Fréchet deformation* of A if (1) $\tilde{A}$ is free as a Fréchet $\mathbb{C}[[h]]$-module, and (2) $\tilde{A}/h\tilde{A} \cong A$ as Fréchet algebras.

We will need the following simple construction. Suppose that $K \to L$ is a homomorphism of commutative Fréchet algebras, and that A is a Fréchet K-algebra. It is easy

to show that the Fréchet L-module $L \widehat{\otimes}_K A$ is a Fréchet L-algebra with multiplication uniquely determined by

$$(\alpha \otimes a)(\beta \otimes b) = \alpha\beta \otimes ab \qquad (\alpha, \beta \in L,\ a, b \in A).$$

Of course, a similar fact holds in the purely algebraic setting (with $\widehat{\otimes}_K$ replaced by $\otimes_K$).

Consider now the Fréchet algebra homomorphism

$$\lambda \colon \mathscr{O}(\mathbb{C}^\times) \to \mathbb{C}[[h]], \quad z \mapsto e^{ih},$$

where $z \in \mathscr{O}(\mathbb{C}^\times)$ is the complex coordinate. We let

$$\tilde{A}(\mathbb{D}_r^n) = \mathbb{C}[[h]] \underset{\mathscr{O}(\mathbb{C}^\times)}{\widehat{\otimes}} \mathscr{O}_{\mathrm{def}}(\mathbb{D}_r^n), \quad \tilde{A}(\mathbb{B}_r^n) = \mathbb{C}[[h]] \underset{\mathscr{O}(\mathbb{C}^\times)}{\widehat{\otimes}} \mathscr{O}_{\mathrm{def}}(\mathbb{B}_r^n).$$

It follows from the above discussion that $\tilde{A}(\mathbb{D}_r^n)$ and $\tilde{A}(\mathbb{B}_r^n)$ are Fréchet $\mathbb{C}[[h]]$-algebras in a canonical way.

Here is the main result of this subsection.

Theorem 8.24. *The Fréchet $\mathbb{C}[[h]]$-algebras $\tilde{A}(\mathbb{D}_r^n)$ and $\tilde{A}(\mathbb{B}_r^n)$ are formal Fréchet deformations of $\mathscr{O}(\mathbb{D}_r^n)$ and $\mathscr{O}(\mathbb{B}_r^n)$, respectively.*

Remark 8.25. The only nontrivial part of Theorem 8.24 is to prove that $\tilde{A}(\mathbb{D}_r^n)$ and $\tilde{A}(\mathbb{B}_r^n)$ are free over $\mathbb{C}[[h]]$. The difficulty comes from the fact that $\mathscr{O}_{\mathrm{def}}(\mathbb{D}_r^n)$ (and presumably $\mathscr{O}_{\mathrm{def}}(\mathbb{B}_r^n)$) are not free over $\mathscr{O}(\mathbb{C}^\times)$ (see Subsection 8.2).

To prove Theorem 8.24, we need some preparation. In what follows, given a Fréchet algebra K and a Fréchet space E, we identify E with a subspace of $K \widehat{\otimes} E$ via the map $x \mapsto 1 \otimes x$.

Lemma 8.26. *Let K be a commutative Fréchet algebra, let E be a Fréchet space, and let M be a Fréchet K-module. Then each continuous bilinear map $E \times E \to M$ uniquely extends to a continuous K-bilinear map $(K \widehat{\otimes} E) \times (K \widehat{\otimes} E) \to M$.*

Proof. Elementary. $\square$

Observe that $\mathscr{O}_{\mathrm{def}}^{\mathrm{reg}}(\mathbb{C}^n)$ contains the Laurent polynomial algebra $\mathscr{O}^{\mathrm{reg}}(\mathbb{C}^\times) = \mathbb{C}[z, z^{-1}]$ (see Lemma 8.4 (i)) and hence is a $\mathscr{O}^{\mathrm{reg}}(\mathbb{C}^\times)$-algebra in a natural way. Consider the $\mathbb{C}[[h]]$-algebra

$$\mathscr{O}_{\mathrm{fdef}}^{\mathrm{reg}}(\mathbb{C}^n) = \mathbb{C}[[h]] \underset{\mathscr{O}^{\mathrm{reg}}(\mathbb{C}^\times)}{\otimes} \mathscr{O}_{\mathrm{def}}^{\mathrm{reg}}(\mathbb{C}^n).$$

By Lemma 8.4 (i), we have a $\mathscr{O}^{\mathrm{reg}}(\mathbb{C}^\times)$-module isomorphism

$$\mathscr{O}^{\mathrm{reg}}(\mathbb{C}^\times) \otimes \mathscr{O}^{\mathrm{reg}}(\mathbb{C}^n) \to \mathscr{O}_{\mathrm{def}}^{\mathrm{reg}}(\mathbb{C}^n), \quad f \otimes x^k \mapsto f x^k \qquad (f \in \mathscr{O}^{\mathrm{reg}}(\mathbb{C}^\times),\ k \in \mathbb{Z}_+^n).$$

Tensoring by $\mathbb{C}[[h]]$, we obtain a $\mathbb{C}[[h]]$-module isomorphism

$$\mathbb{C}[[h]] \otimes \mathscr{O}^{\mathrm{reg}}(\mathbb{C}^n) \to \mathscr{O}_{\mathrm{fdef}}^{\mathrm{reg}}(\mathbb{C}^n), \quad f \otimes x^k \mapsto f x^k \qquad (f \in \mathbb{C}[[h]],\ k \in \mathbb{Z}_+^n). \tag{8.31}$$

Thus $\mathscr{O}_{\mathrm{fdef}}^{\mathrm{reg}}(\mathbb{C}^n)$ is a free $\mathbb{C}[[h]]$-module[1].

Given a Fréchet space E, let $\mathbb{C}[[h; E]]$ denote the vector space of all formal power series with coefficients in E. The isomorphism $\mathbb{C}[[h; E]] \cong E^{\mathbb{Z}_+}$, $\sum_j x_j h^j \mapsto (x_j)$, makes $\mathbb{C}[[h; E]]$ into a Fréchet space. Since the completed projective tensor product commutes

[1]This implies that the h-adic completion of $\mathscr{O}_{\mathrm{fdef}}^{\mathrm{reg}}(\mathbb{C}^n)$ is a formal deformation of $\mathscr{O}^{\mathrm{reg}}(\mathbb{C}^n)$ in the algebraic sense.

with products (see [81, §41.6] or [59, Chap. II, Theorem 5.19]), we have a Fréchet space isomorphism

$$\mathbb{C}[[h]] \mathbin{\widehat{\otimes}} E \cong \mathbb{C}[[h; E]], \quad \Big(\sum_j c_j h^j\Big) \otimes x \mapsto \sum_j c_j x h^j.$$

Proposition 8.27. *Let $D \subset \mathbb{C}^n$ be a complete bounded Reinhardt domain. There exists a unique continuous $\mathbb{C}[[h]]$-bilinear multiplication $\star$ on $\mathbb{C}[[h]] \mathbin{\widehat{\otimes}} \mathscr{O}(D)$ such that for all $j, k = 1, \ldots, n$ we have*

$$x_j \star x_j = \begin{cases} x_j x_k & \text{if } j \le k; \\ e^{-ih} x_k x_j & \text{if } j > k; \end{cases} \tag{8.32}$$

where $x_1, \ldots, x_n$ are the coordinates on $\mathbb{C}^n$. Moreover, $(\mathbb{C}[[h]] \mathbin{\widehat{\otimes}} \mathscr{O}(D), \star)$ is a formal Fréchet deformation of $\mathscr{O}(D)$.

Proof. Let us identify $\mathscr{O}^{\mathrm{reg}}_{\mathrm{fdef}}(\mathbb{C}^n)$ with a dense $\mathbb{C}[[h]]$-submodule of $\mathbb{C}[[h]] \mathbin{\widehat{\otimes}} \mathscr{O}(D)$ via (8.31). Clearly, the multiplication on $\mathscr{O}^{\mathrm{reg}}_{\mathrm{fdef}}(\mathbb{C}^n)$ satisfies (8.32). Thus our goal is to extend the multiplication to $\mathbb{C}[[h]] \mathbin{\widehat{\otimes}} \mathscr{O}(D)$.

Let $f = \sum_k a_k x^k$ and $g = \sum_k b_k x^k \in \mathscr{O}^{\mathrm{reg}}(\mathbb{C}^n) \subset \mathscr{O}^{\mathrm{reg}}_{\mathrm{fdef}}(\mathbb{C}^n)$. As in Theorem 8.23, we have

$$f \star g = \sum_{m \in \mathbb{Z}_+^n} \Bigg(\sum_{k+\ell=m} a_k b_\ell e^{-ih\sigma(\ell,k)} \Bigg) x^m,$$

where σ is given by (8.30). Let now $f = \sum_k a_k x^k$ and $g = \sum_k b_k x^k \in \mathscr{O}(D) \subset \mathbb{C}[[h]] \mathbin{\widehat{\otimes}} \mathscr{O}(D)$. For each $m \in \mathbb{Z}_+^n$, let

$$c_m = c_m(f, g) = \sum_{k+\ell=m} a_k b_\ell e^{-ih\sigma(\ell,k)} \in \mathbb{C}[[h]].$$

We claim that the series $\sum_m c_m x^m$ is absolutely convergent in $\mathbb{C}[[h]] \mathbin{\widehat{\otimes}} \mathscr{O}(D)$. To see this, consider the defining family $\{\|\cdot\|_N : N \in \mathbb{Z}_+\}$ of seminorms on $\mathbb{C}[[h]]$, where

$$\|f\|_N = \sum_{p=0}^N |c_p| \qquad \Big(f = \sum_{p \in \mathbb{Z}_+} c_p h^p \in \mathbb{C}[[h]] \Big). \tag{8.33}$$

Recall from (3.2) that the topology on $\mathscr{O}(D)$ is given by the family $\{\|\cdot\|_\rho : \rho \in (0,1)\}$ of norms, where

$$\|f\|_\rho = \sum_{k \in \mathbb{Z}_+^n} |c_k| \omega_k \rho^{|k|} \qquad \Big(f = \sum_{k \in \mathbb{Z}_+^n} c_k x^k \in \mathscr{O}(D) \Big),$$

and $\omega_k = b_k(D) = \sup_{z \in D} |z^k|$ for all $k \in \mathbb{Z}_+^n$. Therefore the topology on $\mathbb{C}[[h]] \mathbin{\widehat{\otimes}} \mathscr{O}(D)$ is given by the family $\{\|\cdot\|_{N,\rho} : N \in \mathbb{Z}_+,\ \rho \in (0,1)\}$ of seminorms, where $\|\cdot\|_{N,\rho}$ is the projective tensor product of the seminorms $\|\cdot\|_N$ and $\|\cdot\|_\rho$.

Take any $N \in \mathbb{Z}_+$ and $\rho \in (0,1)$. For each $m \in \mathbb{Z}_+^n$ we have

$$\begin{aligned} \|c_m\|_N \le \sum_{k+\ell=m} |a_k b_\ell| \|e^{-ih\sigma(\ell,k)}\|_N &= \sum_{k+\ell=m} \Bigg(|a_k b_\ell| \sum_{p=0}^N \frac{|\sigma(\ell,k)|^p}{p!} \Bigg) \\ &\le \sum_{k+\ell=m} \Bigg(|a_k b_\ell| \sum_{p=0}^N \frac{|k|^p |\ell|^p}{p!} \Bigg) \le C \sum_{k+\ell=m} |a_k b_\ell| |k|^N |\ell|^N, \end{aligned}$$

where $C = \sup_{t \ge 1} t^{-N} \sum_{p=0}^{N} t^p/p!$. Choose $\tau \in (\rho, 1)$, and let

$$C' = C \left(\sup_{t \ge 0} t^N \left(\frac{\rho}{\tau} \right)^t \right)^2 .$$

We have

$$\begin{aligned} \sum_m \|c_m x^m\|_{N,\rho} = \sum_m \|c_m\|_N \|x^m\|_\rho &\le C \sum_m \sum_{k+\ell=m} |a_k b_\ell| |k|^N |\ell|^N \omega_m \rho^{|m|} \\ &\le C \sum_m \sum_{k+\ell=m} |a_k b_\ell| |k|^N |\ell|^N \omega_k \omega_\ell \rho^{|k|} \rho^{|\ell|} \\ &\le C' \sum_m \sum_{k+\ell=m} |a_k b_\ell| \omega_k \omega_\ell \tau^{|k|} \tau^{|\ell|} = C' \|f\|_\tau \|g\|_\tau . \end{aligned}$$

Thus the series $\sum_m c_m x^m$ is absolutely convergent in $\mathbb{C}[[h]] \mathbin{\widehat{\otimes}} \mathscr{O}(D)$, and $\| \sum_m c_m x^m \|_{N,\rho} \le C' \|f\|_\tau \|g\|_\tau$. Hence we have a continuous bilinear map

$$\mathscr{O}(D) \times \mathscr{O}(D) \to \mathbb{C}[[h]] \mathbin{\widehat{\otimes}} \mathscr{O}(D), \quad (f, g) \mapsto f \star g, \tag{8.34}$$

where $f \star g = \sum_m c_m(f,g) x^m$. By Lemma 8.26, (8.34) uniquely extends to a continuous $\mathbb{C}[[h]]$-bilinear map

$$(\mathbb{C}[[h]] \mathbin{\widehat{\otimes}} \mathscr{O}(D)) \times (\mathbb{C}[[h]] \mathbin{\widehat{\otimes}} \mathscr{O}(D)) \to \mathbb{C}[[h]] \mathbin{\widehat{\otimes}} \mathscr{O}(D), \quad (f, g) \mapsto f \star g.$$

By construction, $\star$ extends the multiplication on $\mathscr{O}^{\mathrm{reg}}_{\mathrm{fdef}}(\mathbb{C}^n)$ and hence is associative by continuity. The uniqueness of $\star$ is clear. To show that $\tilde{A} = (\mathbb{C}[[h]] \mathbin{\widehat{\otimes}} \mathscr{O}(D), \star)$ is a formal Fréchet deformation of $\mathscr{O}(D)$, identify $\tilde{A}$ with $\mathbb{C}[[h, \mathscr{O}(D)]]$ as a Fréchet space. The map

$$\tilde{A}/h\tilde{A} \to \mathscr{O}(D), \quad \sum_{p \in \mathbb{Z}_+} c_p h^p \to c_0 \qquad (c_p \in \mathscr{O}(D)), \tag{8.35}$$

is obviously a Fréchet space isomorphism. Since $x_i \star x_j = x_i x_j \mod h\tilde{A}$ for all $i, j = 1, \ldots, n$, we conclude that (8.35) is an algebra isomorphism. ☐

The algebra $(\mathbb{C}[[h]] \mathbin{\widehat{\otimes}} \mathscr{O}(D), \star)$ will be denoted by $\mathscr{O}_{\mathrm{fdef}}(D)$.

Remark 8.28. Proposition 8.27 formally applies to $D = \mathbb{D}^n_r$ and $D = \mathbb{B}^n_r$ only in the case where $r < \infty$. If $r = \infty$, then we let

$$\mathscr{O}_{\mathrm{fdef}}(\mathbb{C}^n) = \varprojlim_{r>0} \mathscr{O}_{\mathrm{fdef}}(\mathbb{D}^n_r) = \varprojlim_{r>0} \mathscr{O}_{\mathrm{fdef}}(\mathbb{B}^n_r).$$

Since the projective tensor product commutes with reduced inverse limits [81, §41.6], we see that $\mathscr{O}_{\mathrm{fdef}}(\mathbb{C}^n) \cong \mathbb{C}[[h]] \mathbin{\widehat{\otimes}} \mathscr{O}(\mathbb{C}^n)$ as a locally convex space, with multiplication uniquely determined by (8.32). Thus $\mathscr{O}_{\mathrm{fdef}}(\mathbb{C}^n)$ is a formal Fréchet deformation of $\mathscr{O}(\mathbb{C}^n)$.

Now, to prove Theorem 8.24, it suffices to construct Fréchet $\mathbb{C}[[h]]$-algebra isomorphisms $\tilde{A}(\mathbb{D}^n_r) \cong \mathscr{O}_{\mathrm{fdef}}(\mathbb{D}^n_r)$ and $\tilde{A}(\mathbb{B}^n_r) \cong \mathscr{O}_{\mathrm{fdef}}(\mathbb{B}^n_r)$. This will be done in the following lemmas.

Lemma 8.29. *There exist Fréchet algebra homomorphisms* $\mathscr{F}(\mathbb{D}^n_r) \to \mathscr{O}_{\mathrm{fdef}}(\mathbb{D}^n_r)$ *and* $\mathscr{F}(\mathbb{B}^n_r) \to \mathscr{O}_{\mathrm{fdef}}(\mathbb{B}^n_r)$ *uniquely determined by* $\zeta_i \mapsto x_i$ $(i = 1, \ldots, n)$.

Proof. Let us start with the (more difficult) case of the homomorphism $\mathscr{F}(\mathbb{B}^n_r) \to \mathscr{O}_{\mathrm{fdef}}(\mathbb{B}^n_r)$. Suppose that $u = \sum_\alpha c_\alpha \zeta_\alpha \in \mathscr{F}(\mathbb{B}^n_r)$. We claim that the family $\sum_\alpha c_\alpha x_\alpha$ is summable in $\mathscr{O}_{\mathrm{fdef}}(\mathbb{B}^n_r)$. To see this, consider the defining family $\{\| \cdot \|_N : N \in \mathbb{Z}_+\}$ of seminorms on $\mathbb{C}[[h]]$ given by (8.33) and the defining family $\{\| \cdot \|_{\mathbb{B},\rho} : \rho \in (0, r)\}$ of norms on $\mathscr{O}(\mathbb{B}^n_r)$

given by (3.9). The projective tensor product of the seminorms $\|\cdot\|_N$ and $\|\cdot\|_{\mathbb{B},\rho}$ will be denoted by $\|\cdot\|_{N,\rho}$. We have

$$\begin{aligned}
\sum_{k\in\mathbb{Z}_+^n}\Big\|\sum_{\alpha\in\mathrm{p}^{-1}(k)} c_\alpha x_\alpha\Big\|_{N,\rho} &= \sum_{k\in\mathbb{Z}_+^n}\Big\|\sum_{\alpha\in\mathrm{p}^{-1}(k)} c_\alpha e^{-\mathrm{im}(\alpha)h} x^k\Big\|_{N,\rho}\\
&= \sum_{k\in\mathbb{Z}_+^n}\Big\|\sum_{\alpha\in\mathrm{p}^{-1}(k)} c_\alpha e^{-\mathrm{im}(\alpha)h}\Big\|_N \left(\frac{k!}{|k|!}\right)^{1/2}\rho^{|k|}\\
&= \sum_{k\in\mathbb{Z}_+^n}\sum_{j=0}^N\Big|\sum_{\alpha\in\mathrm{p}^{-1}(k)} c_\alpha \frac{(-i\mathrm{m}(\alpha))^j}{j!}\Big|\left(\frac{k!}{|k|!}\right)^{1/2}\rho^{|k|}\\
&\le \sum_{k\in\mathbb{Z}_+^n}\sum_{\alpha\in\mathrm{p}^{-1}(k)}|c_\alpha|\left(\sum_{j=0}^N\frac{\mathrm{m}(\alpha)^j}{j!}\right)\left(\frac{k!}{|k|!}\right)^{1/2}\rho^{|k|}\\
&\le \sum_{k\in\mathbb{Z}_+^n}\sum_{\alpha\in\mathrm{p}^{-1}(k)}|c_\alpha|\left(\sum_{j=0}^N\frac{|\alpha|^{2j}}{j!}\right)\left(\frac{k!}{|k|!}\right)^{1/2}\rho^{|k|}. \qquad (8.36)
\end{aligned}$$

Here we have used the obvious inequality $\mathrm{m}(\alpha)\le|\alpha|^2$. Now choose $\rho_1\in(\rho,r)$, and find $C_1,C_2>0$ such that

$$\begin{aligned}
&\sum_{j=0}^N\frac{t^{2j}}{j!}\le C_1(1+t^{2N}) \qquad (t\ge 0);\\
&C_1(1+t^{2N})\rho^t\le C_2\rho_1^t \qquad (t\ge 0).
\end{aligned}$$

Since $|\alpha|=|k|$ whenever $\alpha\in\mathrm{p}^{-1}(k)$, we can continue (8.36) as follows:

$$\begin{aligned}
\sum_{k\in\mathbb{Z}_+^n}\sum_{\alpha\in\mathrm{p}^{-1}(k)}|c_\alpha|\left(\sum_{j=0}^N\frac{|\alpha|^{2j}}{j!}\right)\left(\frac{k!}{|k|!}\right)^{1/2}\rho^{|k|} &\le C_1\sum_{k\in\mathbb{Z}_+^n}\sum_{\alpha\in\mathrm{p}^{-1}(k)}|c_\alpha|(1+|k|^{2N})\left(\frac{k!}{|k|!}\right)^{1/2}\rho^{|k|}\\
&\le C_2\sum_{k\in\mathbb{Z}_+^n}\sum_{\alpha\in\mathrm{p}^{-1}(k)}|c_\alpha|\left(\frac{k!}{|k|!}\right)^{1/2}\rho_1^{|k|}. \qquad (8.37)
\end{aligned}$$

Finally, applying the Cauchy-Bunyakowsky-Schwarz inequality together with (4.6), we continue (8.37) as follows:

$$\begin{aligned}
C_2\sum_{k\in\mathbb{Z}_+^n}\sum_{\alpha\in\mathrm{p}^{-1}(k)}|c_\alpha|\left(\frac{k!}{|k|!}\right)^{1/2}\rho_1^{|k|} &\le C_2\sum_{k\in\mathbb{Z}_+^n}\left(\sum_{\alpha\in\mathrm{p}^{-1}(k)}|c_\alpha|^2\right)^{1/2}\rho_1^{|k|}\\
&= C_2\Big\|\sum_\alpha c_\alpha\zeta_\alpha\Big\|_{\rho_1}^\circ.
\end{aligned} \qquad (8.38)$$

Thus we see that for each $u=\sum_\alpha c_\alpha\zeta_\alpha\in\mathscr{F}(\mathbb{B}_r^n)$ the family $\sum_\alpha c_\alpha x_\alpha$ is summable in $\mathscr{O}_{\mathrm{fdef}}(\mathbb{B}_r^n)$. Letting $\varphi(u)=\sum_\alpha c_\alpha x_\alpha$, we obtain a linear map $\varphi\colon\mathscr{F}(\mathbb{B}_r^n)\to\mathscr{O}_{\mathrm{fdef}}(\mathbb{B}_r^n)$ such that $\varphi(\zeta_i)=x_i$ for all $i=1,\dots,n$. Moreover, (8.36)–(8.38) imply that $\|\varphi(u)\|_{N,\rho}\le C_2\|u\|_{\rho_1}^\circ$, so that φ is continuous. Since the restriction of φ to the dense subalgebra $F_n\subset\mathscr{F}(\mathbb{B}_r^n)$ is clearly an algebra homomorphism, we conclude that φ is a homomorphism.

The homomorphism $\mathscr{F}(\mathbb{D}^n_r) \to \mathscr{O}_{\mathrm{fdef}}(\mathbb{D}^n_r)$ is constructed similarly. Specifically, we replace $\|\cdot\|_{\mathbb{B},\rho}$ by $\|\cdot\|_{\mathbb{D},\rho}$ in the above argument, remove the factor $\bigl(\frac{k!}{|k|!}\bigr)^{1/2}$ from (8.36) and (8.37), and observe that the last expression in (8.37) will be exactly $C_2\|\sum_\alpha c_\alpha\zeta_\alpha\|_{\rho_1,1}$, where the norm $\|\cdot\|_{\rho_1,1}$ is given by (6.2). Thus (8.38) is not needed in this case. The last step of the construction is identical to the case of $\mathbb{B}^n_r$. $\square$

Remark 8.30. It follows from the above proof that Lemma 8.29 holds with $\mathscr{F}(\mathbb{D}^n_r)$ replaced by $\mathscr{F}^{\mathrm{T}}(\mathbb{D}^n_r)$.

Remark 8.31. If we knew that $\mathscr{O}_{\mathrm{fdef}}(\mathbb{D}^n_r)$ is an Arens-Michael algebra, then the construction of the homomorphism $\mathscr{F}(\mathbb{D}^n_r) \to \mathscr{O}_{\mathrm{fdef}}(\mathbb{D}^n_r)$ could easily be deduced from the universal property of $\mathscr{F}(\mathbb{D}^n_r)$. As a matter of fact, $\mathscr{O}_{\mathrm{fdef}}(D)$ is indeed an Arens-Michael algebra (for each complete bounded Reinhardt domain D, as well as for $D = \mathbb{C}^n$), but the direct proof of this result is rather technical, so we omit it. Anyway, the fact that $\mathscr{O}_{\mathrm{fdef}}(\mathbb{D}^n_r)$ and $\mathscr{O}_{\mathrm{fdef}}(\mathbb{B}^n_r)$ are Arens-Michael algebras will be immediate from Theorem 8.24.

From now on, the image of $x_j \in \mathscr{O}_{\mathrm{def}}(\mathbb{D}^n_r)$ $(j = 1, \ldots, n)$ under the map

$$\mathscr{O}_{\mathrm{def}}(\mathbb{D}^n_r) \to \tilde{A}(\mathbb{D}^n_r) = \mathbb{C}[[h]] \underset{\mathscr{O}(\mathbb{C}^\times)}{\widehat{\otimes}} \mathscr{O}_{\mathrm{def}}(\mathbb{D}^n_r), \quad u \mapsto 1_{\mathbb{C}[[h]]} \otimes u,$$

will be denoted by the same symbol x_j. This will not lead to a confusion. The same convention applies to $\mathscr{O}_{\mathrm{def}}(\mathbb{B}^n_r)$ and $\tilde{A}(\mathbb{B}^n_r)$.

Lemma 8.32. *There exist continuous linear maps $\mathscr{O}(\mathbb{D}^n_r) \to \tilde{A}(\mathbb{D}^n_r)$ and $\mathscr{O}(\mathbb{B}^n_r) \to \tilde{A}(\mathbb{B}^n_r)$ uniquely determined by $x^k \mapsto x^k$ $(k \in \mathbb{Z}^n_+)$.*

Proof. As in Lemma 8.29, we start by constructing the map $\mathscr{O}(\mathbb{B}^n_r) \to \tilde{A}(\mathbb{B}^n_r)$ (as we shall see, the case of $\mathbb{D}^n_r$ is much easier). Applying the functor $\mathbb{C}[[h]] \widehat{\otimes}_{\mathscr{O}(\mathbb{C}^\times)}(-)$ to the quotient map

$$\mathscr{O}(\mathbb{C}^\times) \widehat{\otimes} \mathscr{F}(\mathbb{B}^n_r) \to \mathscr{O}_{\mathrm{def}}(\mathbb{B}^n_r),$$

we obtain a surjective homomorphism

$$\tilde{\pi}_{\mathbb{B}} \colon \mathbb{C}[[h]] \widehat{\otimes} \mathscr{F}(\mathbb{B}^n_r) = \mathbb{C}[[h; \mathscr{F}(\mathbb{B}^n_r)]] \to \tilde{A}(\mathbb{B}^n_r)$$

of Fréchet $\mathbb{C}[[h]]$-algebras. For each $s \in \mathbb{Z}_+$ and each $\rho \in (0, r)$, define a seminorm $\|\cdot\|_{s,\rho}$ on $\mathbb{C}[[h; \mathscr{F}(\mathbb{B}^n_r)]]$ by

$$\Bigl\|\sum_j f_j h^j\Bigr\|_{s,\rho} = \|f_s\|^\circ_\rho \qquad (f_j \in \mathscr{F}(\mathbb{B}^n_r)).$$

Clearly, $\{\|\cdot\|_{s,\rho} : s \in \mathbb{Z}_+,\ \rho \in (0, r)\}$ is a (nondirected) defining family of seminorms on $\mathbb{C}[[h; \mathscr{F}(\mathbb{B}^n_r)]]$.

Given $k \in \mathbb{Z}^n_+$, let

$$u_k = \frac{k!}{|k|!} \sum_{\alpha \in \mathrm{p}^{-1}(k)} e^{\mathrm{i}m(\alpha)h} \zeta_\alpha \in \mathbb{C}[[h, \mathscr{F}(\mathbb{B}^n_r)]].$$

Observe that

$$\tilde{\pi}_{\mathbb{B}}(u_k) = \frac{k!}{|k|!} \sum_{\alpha \in \mathrm{p}^{-1}(k)} e^{\mathrm{i}m(\alpha)h} x_\alpha = \frac{k!}{|k|!} \sum_{\alpha \in \mathrm{p}^{-1}(k)} x^k = x^k. \tag{8.39}$$

Take $s \in \mathbb{Z}_+$ and $\rho \in (0, r)$, choose any $\rho_1 \in (\rho, r)$, and find $C > 0$ such that

$$t^{2s}\rho^t \le C\rho_1^t \qquad (t \ge 0).$$

Using the fact that $\mathrm{m}(\alpha) \le |\alpha|^2$, we obtain

$$\|u_k\|_{s,\rho} = \frac{k!}{|k|!} \Big\| \sum_{\alpha\in \mathrm{p}^{-1}(k)} \frac{(i\mathrm{m}(\alpha))^s}{s!}\, \zeta_\alpha \Big\|_\rho^\circ = \frac{k!}{|k|!} \Big(\sum_{\alpha\in \mathrm{p}^{-1}(k)} \frac{\mathrm{m}(\alpha)^{2s}}{(s!)^2} \Big)^{1/2} \rho^{|k|}$$
$$\le |k|^{2s} \left(\frac{k!}{|k|!}\right)^{1/2} \rho^{|k|} \le C \left(\frac{k!}{|k|!}\right)^{1/2} \rho_1^{|k|} = C\|x^k\|_{\mathbb{B},\rho_1}.$$

This implies that for each $f = \sum_k c_k x^k \in \mathscr{O}(\mathbb{B}_r^n)$ the series $\sum_k c_k u_k$ absolutely converges in $\mathbb{C}[[h;\mathscr{F}(\mathbb{B}_r^n)]]$. Moreover, we have

$$\Big\| \sum_k c_k u_k \Big\|_{s,\rho} \le C\|f\|_{\mathbb{B},\rho_1}.$$

Hence we obtain a continuous linear map

$$\psi\colon \mathscr{O}(\mathbb{B}_r^n) \to \mathbb{C}[[h;\mathscr{F}(\mathbb{B}_r^n)]], \quad x^k \mapsto u_k \qquad (k\in\mathbb{Z}_+^n).$$

Taking into account (8.39), we conclude that $\tilde\pi_{\mathbb{B}}\psi\colon \mathscr{O}(\mathbb{B}_r^n) \to \tilde A(\mathbb{B}_r^n)$ is the map we are looking for.

To construct the map $\mathscr{O}(\mathbb{D}_r^n) \to \tilde A(\mathbb{D}_r^n)$, observe that for each $k\in\mathbb{Z}_+^n$, each $\rho\in(0,r)$, and each $\tau\ge 1$ we have

$$\|\zeta^k\|_{\rho,\tau} \le \tau^n \rho^{|k|} = \tau^n\|x^k\|_{\mathbb{D},\rho}$$

(where the norm $\|\cdot\|_{\rho,\tau}$ on $\mathscr{F}(\mathbb{D}_r^n)$ is given by (6.2)). This implies that for each $f = \sum_k c_k x^k \in \mathscr{O}(\mathbb{D}_r^n)$ the series $\sum_k c_k\zeta^k$ absolutely converges in $\mathscr{F}(\mathbb{D}_r^n)$ to an element $j(f)$, and that $j\colon \mathscr{O}(\mathbb{D}_r^n) \to \mathscr{F}(\mathbb{D}_r^n)$ is a continuous linear map. Clearly, the composition

$$\mathscr{O}(\mathbb{D}_r^n) \xrightarrow{j} \mathscr{F}(\mathbb{D}_r^n) \hookrightarrow \mathbb{C}[[h;\mathscr{F}(\mathbb{D}_r^n)]] \xrightarrow{\tilde\pi_{\mathbb{D}}} \tilde A(\mathbb{D}_r^n)$$

(where $\tilde\pi_{\mathbb{D}}$ is defined similarly to $\tilde\pi_{\mathbb{B}}$) is the map we are looking for. ☐

Theorem 8.33. *There exist Fréchet $\mathbb{C}[[h]]$-algebra isomorphisms $\tilde A(\mathbb{D}_r^n) \to \mathscr{O}_{\mathrm{fdef}}(\mathbb{D}_r^n)$ and $\tilde A(\mathbb{B}_r^n) \to \mathscr{O}_{\mathrm{fdef}}(\mathbb{B}_r^n)$ uniquely determined by $x_i \mapsto x_i$ $(i=1,\dots,n)$.*

Proof. Let $D=\mathbb{D}_r^n$ or $D=\mathbb{B}_r^n$, and let

$$\varphi_0\colon \mathscr{F}(D) \to \mathscr{O}_{\mathrm{fdef}}(D), \quad \zeta_i \mapsto x_i \qquad (i=1,\dots,n)$$

be the Fréchet algebra homomorphism constructed in Lemma 8.29. Consider the map

$$\varphi_1\colon \mathscr{O}(\mathbb{C}^\times) \mathbin{\widehat\otimes} \mathscr{F}(D) \to \mathscr{O}_{\mathrm{fdef}}(D), \quad f\otimes u \mapsto \lambda(f)\varphi_0(u).$$

Clearly, φ_1 is a Fréchet $\mathscr{O}(\mathbb{C}^\times)$-algebra homomorphism, and φ_1 vanishes on I_D. Hence φ_1 induces a Fréchet $\mathscr{O}(\mathbb{C}^\times)$-algebra homomorphism

$$\varphi_2\colon \mathscr{O}_{\mathrm{def}}(D) \to \mathscr{O}_{\mathrm{fdef}}(D), \quad f\otimes u + I_D \mapsto \lambda(f)\varphi_0(u).$$

Consider now the map

$$\varphi\colon \tilde A(D) = \mathbb{C}[[h]] \mathbin{\widehat{\otimes}}_{\mathscr{O}(\mathbb{C}^\times)} \mathscr{O}_{\mathrm{def}}(D) \to \mathscr{O}_{\mathrm{fdef}}(D), \quad g\otimes v \mapsto g\varphi_2(v).$$

Clearly, φ takes x_i to x_i and is a Fréchet $\mathbb{C}[[h]]$-algebra homomorphism. We claim that φ is an isomorphism. To see this, let

$$\psi_0\colon \mathscr{O}(D) \to \tilde A(D), \quad x^k \mapsto x^k \qquad (k\in\mathbb{Z}_+^n)$$

be the linear map constructed in Lemma 8.32. We have a Fréchet $\mathbb{C}[[h]]$-module morphism

$$\psi\colon \mathscr{O}_{\mathrm{fdef}}(D) = \mathbb{C}[[h]] \mathbin{\widehat{\otimes}} \mathscr{O}(D) \to \tilde{A}(D), \quad f \otimes g \mapsto f\psi_0(g).$$

Clearly, $(\varphi\psi)(x^k) = x^k$ and $(\psi\varphi)(x^k) = x^k$ for all $k \in \mathbb{Z}_+^n$. Since the $\mathbb{C}[[h]]$-submodule generated by $\{x^k : k \in \mathbb{Z}_+^n\}$ is dense both in $\mathscr{O}_{\mathrm{fdef}}(D)$ and in $\tilde{A}(D)$, we conclude that $\varphi\psi = 1_{\mathscr{O}_{\mathrm{fdef}}(D)}$ and $\psi\varphi = 1_{\tilde{A}(D)}$. Hence φ is a Fréchet $\mathbb{C}[[h]]$-algebra isomorphism. $\square$

Now Theorem 8.24 is immediate from Theorem 8.33 and Proposition 8.27.

Appendix A. Locally convex bundles

In this Appendix, we collect some facts on bundles of locally convex spaces and algebras. Most definitions below are modifications of those contained in [53] (cf. also [35, 170]). The principal difference between our approach and the approach of [53] is that we do not endow the total space E of a locally convex bundle with a family of seminorms. Instead, we introduce a coarser "locally convex uniform vector structure" on E compatible with the topology on E (see Definitions A.3 and A.10). The reason is that we need a functor from locally convex K-modules (where K is a subalgebra of $C(X)$) to locally convex bundles over X (see Theorems A.28 and A.31). It seems that the approach of [53] is not appropriate for this purpose.

By a *family of vector spaces* over a set X we mean a pair (E, p), where E is a set and $p\colon E \to X$ is a surjective map, together with a vector space structure on each fiber $E_x = p^{-1}(x)$ ($x \in X$). As usual, we let

$$E \times_X E = \{(u, v) \in E \times E : p(u) = p(v)\}.$$

For a subset $V \subset X$, a map $s\colon V \to E$ is a *section* of (E, p) over V if $ps = 1_V$. It will be convenient to denote the value of s at $x \in V$ by s_x. The vector space of all sections of (E, p) over V will be denoted by $S(V, E)$. In other words, $S(V, E) = \prod_{x \in V} E_x$.

Definition A.1. Let X be a topological space. By a *prebundle of topological vector spaces* over X we mean a family (E, p) of vector spaces over X together with a topology on E such that p is continuous and open, the zero section $0\colon X \to E$ is continuous, and the operations

$$E \times_X E \to E, \quad (u, v) \mapsto u + v,$$
$$\mathbb{C} \times E \to E, \quad (\lambda, v) \mapsto \lambda v,$$

are also continuous.

Remark A.2. If (E, p) is a prebundle of topological vector spaces, then each fiber E_x is clearly a topological vector space with respect to the topology inherited from E.

If (E, p) is a prebundle of topological vector spaces over X, then the set of all continuous sections of (E, p) over $V \subset X$ will be denoted by $\Gamma(V, E)$. The above axioms readily imply that $\Gamma(V, E)$ is a vector subspace of $S(V, E)$.

Prebundles of topological vector spaces are too general objects for our purposes. First, we would like that the topology on each fiber E_x inherited from E be locally convex. What is more important, we would like to consider prebundles endowed with an additional structure that would enable us to "compare" 0-neighborhoods in different fibers. This can be achieved as follows.

Let (E,p) be a family of vector spaces over a set X. We say that a subset $S \subset E$ is *absolutely convex* (respectively, *absorbing*) if $S \cap E_x$ is absolutely convex (respectively, absorbing) in E_x, for each $x \in X$.

Definition A.3. Let X be a set, and let (E,p) be a family of vector spaces over X. By a *locally convex uniform vector structure* on E we mean a family $\mathscr{U}$ of subsets of E satisfying the following conditions:

(U1) Each $U \in \mathscr{U}$ is absorbing and absolutely convex.
(U2) If $U \in \mathscr{U}$ and $\lambda \in \mathbb{C} \setminus \{0\}$, then $\lambda U \in \mathscr{U}$.
(U3) If $U, V \in \mathscr{U}$, then $U \cap V \in \mathscr{U}$.
(U4) If $U \in \mathscr{U}$ and $V \supset U$ is an absolutely convex subset of E, then $V \in \mathscr{U}$.

Example A.4. If X is a single point and E is just a vector space, then there is a 1-1 correspondence between locally convex topologies on E (i.e., topologies on E making E into a locally convex topological vector space) and locally convex uniform vector structures on E. Indeed, if E is endowed with a locally convex topology, then the family of all absolutely convex 0-neighborhoods in E is a locally convex uniform vector structure on E. Moreover, each locally convex uniform vector structure arises in this way (see, e.g., [80, §18.1]).

Remark A.5. A locally convex uniform vector structure $\mathscr{U}$ on E determines a uniform structure $\tilde{\mathscr{U}}$ (in Weil's sense; see, e.g., [48, Chap. 8]) on E whose basis consists of all sets of the form
$$\tilde{U} = \bigl\{(u,v) \in E \times_X E : u - v \in U\bigr\} \qquad (U \in \mathscr{U}).$$

Definition A.6. Let X be a set, (E,p) be a family of vector spaces over X, and $\mathscr{U}$ be a locally convex uniform vector structure on E. We say that a subfamily $\mathscr{B} \subset \mathscr{U}$ is a *base* of $\mathscr{U}$ if for each $U \in \mathscr{U}$ there exists $B \in \mathscr{B}$ such that $B \subset U$.

Clearly, a filter base $\mathscr{B} \subset 2^E$ consisting of absorbing, absolutely convex subsets of E is a base of a (necessarily unique) locally convex uniform vector structure if and only if $\mathscr{B}$ satisfies the condition

(BU) For each $B \in \mathscr{B}$ there exists $B' \in \mathscr{B}$ such that $B' \subset (1/2)B$.

It is a standard fact that each locally convex topology on a vector space is generated by a family of seminorms. This result can easily be extended to locally convex uniform vector structures on a family of vector spaces. Let (E,p) be a family of vector spaces over a set X. By definition, a function $\|\cdot\| \colon E \to [0,+\infty)$ is a *seminorm* if the restriction of $\|\cdot\|$ to each fiber is a seminorm in the usual sense. Suppose that $\mathscr{N} = \{\|\cdot\|_\lambda : \lambda \in \Lambda\}$ is a family of seminorms on E. We assume that $\mathscr{N}$ is *directed*, that is, for each $\lambda, \mu \in \Lambda$ there exist $C > 0$ and $\nu \in \Lambda$ such that $\|\cdot\|_\lambda \le C\|\cdot\|_\nu$ and $\|\cdot\|_\mu \le C\|\cdot\|_\nu$. Given $\lambda \in \Lambda$ and $\varepsilon > 0$, let $\mathsf{U}_{\lambda,\varepsilon} = \{u \in E : \|u\|_\lambda < \varepsilon\}$.

Proposition A.7. *Let X be a set, and let (E,p) be a family of vector spaces over X.*

(i) *For each directed family $\mathscr{N} = \{\|\cdot\|_\lambda : \lambda \in \Lambda\}$ of seminorms on E, the family $\mathscr{B}_{\mathscr{N}} = \{\mathsf{U}_{\lambda,\varepsilon} : \lambda \in \Lambda,\ \varepsilon > 0\}$ is a base of a locally convex uniform vector structure $\mathscr{U}_{\mathscr{N}}$ on E.*
(ii) *Conversely, for each locally convex uniform vector structure $\mathscr{U}$ on E there exists a directed family $\mathscr{N}$ of seminorms on E such that $\mathscr{U} = \mathscr{U}_{\mathscr{N}}$. Specifically, we can take*

$\mathscr{N} = \{p_B : B \in \mathscr{B}\}$, *where* $\mathscr{B}$ *is any base of* $\mathscr{U}$ *and* p_B *is the Minkowski functional of* B.

Proof. (i) Clearly, $\mathscr{B}_{\mathscr{N}}$ is a filter base on E, each set belonging to $\mathscr{B}_{\mathscr{N}}$ is absorbing and absolutely convex, and $\mathscr{B}_{\mathscr{N}}$ satisfies (BU).

(ii) Given $B_1, B_2 \in \mathscr{B}$, find $B_3 \in \mathscr{B}$ such that $B_3 \subset B_1 \cap B_2$. We clearly have $\max\{p_{B_1}, p_{B_2}\} \le p_{B_3}$, which implies that $\mathscr{N}$ is directed. It is a standard fact that for each absorbing, absolutely convex set B we have

$$\{u \in E : p_B(u) < 1\} \subset B \subset \{u \in E : p_B(u) \le 1\}.$$

Thus for each $B \in \mathscr{B}$ we have $\mathsf{U}_{p_B,1} \subset B$, whence $\mathscr{U} \subset \mathscr{U}_{\mathscr{N}}$. On the other hand, for each $B \in \mathscr{B}$ and each $\varepsilon > 0$ we have $(\varepsilon/2)B \subset \mathsf{U}_{p_B,\varepsilon}$, whence $\mathscr{U}_{\mathscr{N}} \subset \mathscr{U}$. □

Definition A.8. If $\mathscr{N} = \{\|\cdot\|_\lambda : \lambda \in \Lambda\}$ and $\mathscr{N}' = \{\|\cdot\|_\mu : \mu \in \Lambda'\}$ are two directed families of seminorms on E, then we say that $\mathscr{N}$ is *dominated* by $\mathscr{N}'$ (and write $\mathscr{N} \prec \mathscr{N}'$) if for each $\lambda \in \Lambda$ there exist $C > 0$ and $\mu \in \Lambda'$ such that $\|\cdot\|_\lambda \le C\|\cdot\|_\mu$. If $\mathscr{N} \prec \mathscr{N}'$ and $\mathscr{N}' \prec \mathscr{N}$, then we say that $\mathscr{N}$ and $\mathscr{N}'$ are *equivalent* and write $\mathscr{N} \sim \mathscr{N}'$.

A standard argument shows that $\mathscr{N} \prec \mathscr{N}'$ if and only if $\mathscr{U}_{\mathscr{N}} \subset \mathscr{U}_{\mathscr{N}'}$, and hence $\mathscr{N} \sim \mathscr{N}'$ if and only if $\mathscr{U}_{\mathscr{N}} = \mathscr{U}_{\mathscr{N}'}$.

Given a family (E, p) of vector spaces over X, let

$$\mathrm{add}\colon E \times_X E \to E, \quad (u, v) \mapsto u + v.$$

For each pair S, T of subsets of E, we let $S + T = \mathrm{add}((S \times T) \cap (E \times_X E))$.

Lemma A.9. *Let X be a topological space, and let (E, p) be a prebundle of topological vector spaces over X. For each open set $V \subset X$, each $s \in \Gamma(V, E)$, and each open set $U \subset E$, the set $s(V) + U$ is open in E.*

Proof. Clearly, the map $p^{-1}(V) \to E$, $v \mapsto v - s_{p(v)}$, is continuous. Now the result follows from the equality

$$s(V) + U = \big\{v \in p^{-1}(V) : v - s_{p(v)} \in U\big\}.$$ □

Definition A.10. Let X be a topological space. A *bundle of locally convex spaces* (or a *locally convex bundle*) over X is a triple $(E, p, \mathscr{U})$, where (E, p) is a prebundle of topological vector spaces over X and $\mathscr{U}$ is a locally convex uniform vector structure on E satisfying the following compatibility axioms:

(B0) For each $U \in \mathscr{U}$ there exists an open subset $U_0 \subset U$, $U_0 \in \mathscr{U}$.

(B1) The family

$$\big\{s(V) + U : V \subset X \text{ open}, \ s \in \Gamma(V, E), \ U \in \mathscr{U}, \ U \text{ is open}\big\}$$

is a base for the topology on E.

(B2) For each $x \in X$, the set

$$\big\{s_x : s \in \Gamma(V, E), \ V \text{ is an open neighborhood of } x\big\}$$

is dense in E_x with respect to the topology inherited from E.

If, in addition, each fiber E_x is a Fréchet space with respect to the topology inherited from E, then we say that $(E, p, \mathscr{U})$ is a *Fréchet space bundle*.

Remark A.11. We would like to stress that the topology on E does not coincide with the topology $\tau(\mathscr{U})$ determined by the uniform structure $\widetilde{\mathscr{U}}$ (see Remark A.5). In fact, each fiber E_x is $\tau(\mathscr{U})$-open, which is not the case for the original topology on E (unless X is discrete). Axiom (B1) implies that $\tau(\mathscr{U})$ is stronger than the original topology on E.

Lemma A.12. *Let X be a topological space, and let $(E, p, \mathscr{U})$ be a locally convex bundle over X. Suppose that $\mathscr{B}$ is a base of $\mathscr{U}$ consisting of open sets. Then*

(i) *the family*
$$\{s(V) + B : V \subset X \text{ open}, \ s \in \Gamma(V, E), \ B \in \mathscr{B}\}$$
is a base for the topology on E;

(ii) *for each open set $X' \subset X$, each $s \in \Gamma(X', E)$, and each $x \in X'$, the family*
$$\{s(V) + B : V \subset X' \text{ is an open neighborhood of } x, \ B \in \mathscr{B}\}$$
is a base of open neighborhoods of s_x.

Proof. Let W be an open subset of E, and let $u \in W$. Without loss of generality, we may assume that $W = s(V) + U$ for some open set $V \subset X$, $s \in \Gamma(V, E)$, and an open set $U \in \mathscr{U}$. Letting $x = p(u)$, we see that $u - s_x \in U$. Since U is open, there exists $\delta \in (0, 1)$ such that $u - s_x \in (1 - 2\delta)U$. Choose $B \in \mathscr{B}$ such that $B \subset \delta U$. By Remark A.2, $u + (B \cap E_x)$ is a neighborhood of u in E_x. Hence (B2) implies that there exist an open neighborhood $V' \subset V$ of x and $t \in \Gamma(V', E)$ such that $t_x \in u + B$. We have
$$t_x - s_x = (t_x - u) + (u - s_x) \in \delta U + (1 - 2\delta)U = (1 - \delta)U.$$
Hence there exists an open neighborhood $V'' \subset V'$ of x such that $(t - s)(V'') \subset (1 - \delta)U$. We clearly have $u \in t(V'') + B$, and
$$t(V'') + B \subset s(V'') + (t - s)(V'') + B \subset s(V) + (1 - \delta)U + \delta U = s(V) + U.$$
This completes the proof of (i). The proof of (ii) is similar and is therefore omitted (cf. also [53]). ☐

Let us now study relations between locally convex bundles in the sense of Definition A.10 and locally convex bundles in the sense of [53]. Let (E, p) be a family of vector spaces over a set X, and let $\mathscr{N} = \{\|\cdot\|_\lambda : \lambda \in \Lambda\}$ be a directed family of seminorms on E. Given $\lambda \in \Lambda$, $\varepsilon > 0$, a set $V \subset X$, and a section $s \colon V \to E$, define the "ε-tube" around s by
$$\mathsf{T}(V, s, \lambda, \varepsilon) = \{v \in E : p(v) \in V, \ \|v - s_{p(v)}\|_\lambda < \varepsilon\}.$$
Observe that $\mathsf{T}(X, 0, \lambda, \varepsilon) = \mathsf{U}_{\lambda,\varepsilon}$ and
$$\mathsf{T}(V, s, \lambda, \varepsilon) = s(V) + \mathsf{U}_{\lambda,\varepsilon}. \tag{A.1}$$

Definition A.13. Let X be a topological space, (E, p) be a prebundle of topological vector spaces over X, and $\mathscr{N} = \{\|\cdot\|_\lambda : \lambda \in \Lambda\}$ be a directed family of seminorms on E. We say that $\mathscr{N}$ is *admissible* if the following conditions hold (cf. [6, 53, 64, 169]):

(Ad1) The family
$$\mathscr{B}(\mathscr{N}) = \{\mathsf{T}(V, s, \lambda, \varepsilon) : V \subset X \text{ open}, \ s \in \Gamma(V, E), \ \lambda \in \Lambda, \ \varepsilon > 0\}$$
consists of open sets and is a base for the topology on E.

(Ad2) For each $x \in X$, the set

$$\{s_x : s \in \Gamma(V, E),\ V \text{ is an open neighborhood of } x\}$$

is dense in E_x with respect to the topology generated by the restrictions of the seminorms $\|\cdot\|_\lambda$ ($\lambda \in \Lambda$) to E_x.

Remark A.14. The condition that all sets belonging to $\mathscr{B}(\mathscr{N})$ are open is equivalent to the upper semicontinuity of all seminorms from $\mathscr{N}$. Indeed, $\|\cdot\|_\lambda$ is upper semicontinuous if and only if $\mathsf{U}_{\lambda,\varepsilon} = \mathsf{T}(X, 0, \lambda, \varepsilon)$ is open for all $\varepsilon > 0$. By (A.1) and Lemma A.9, this implies that each $\mathsf{T}(V, s, \lambda, \varepsilon) \in \mathscr{B}(\mathscr{N})$ is open.

Remark A.15. Some authors (e.g., [6,53]) use a stronger form of (Ad1) and (Ad2) in which $\Gamma(V, E)$ is replaced by the space $\Gamma_{\mathscr{N}}(V, E)$ of $\mathscr{N}$*-bounded* sections (i.e., those $s \in \Gamma(V, E)$ for which the function $x \mapsto \|s(x)\|_\lambda$ is bounded for every $\lambda \in \Lambda$). This restriction is not needed for our purposes. Anyway, the stronger form of (Ad1) and (Ad2) is clearly equivalent to ours in the case where X is locally compact.

Lemma A.16. *Let (E, p) be a prebundle of topological vector spaces over X, and let $\mathscr{N}$ and $\mathscr{N}'$ be directed families of upper semicontinuous seminorms on E.*

(i) *If $\mathscr{N} \prec \mathscr{N}'$ and $\mathscr{N}'$ satisfies* (Ad2), *then each set from $\mathscr{B}(\mathscr{N})$ is a union of sets belonging to $\mathscr{B}(\mathscr{N}')$.*
(ii) *If $\mathscr{N} \sim \mathscr{N}'$ and $\mathscr{N}$ is admissible, then so is $\mathscr{N}'$.*

Proof. (i) Let $\mathscr{N} = \{\|\cdot\|_\lambda : \lambda \in \Lambda\}$ and $\mathscr{N}' = \{\|\cdot\|_\mu : \mu \in \Lambda'\}$. Take any $\mathsf{T}(V, s, \lambda, \varepsilon) \in \mathscr{B}(\mathscr{N})$, and let $u \in \mathsf{T}(V, s, \lambda, \varepsilon)$. Choose $\mu \in \Lambda'$ and $C > 0$ such that $\|\cdot\|_\lambda \le C\|\cdot\|_\mu$. Let $x = p(u)$, and find $\delta > 0$ such that

$$2C\delta + \|u - s_x\|_\lambda < \varepsilon.$$

Since $\mathscr{N}'$ satisfies (Ad2), there exist an open neighborhood W of x and $t \in \Gamma(W, E)$ such that $\|t_x - u\|_\mu < \delta$. We have

$$\|t_x - s_x\|_\lambda \le \|t_x - u\|_\lambda + \|u - s_x\|_\lambda \le C\|t_x - u\|_\mu + \|u - s_x\|_\lambda < C\delta + \|u - s_x\|_\lambda.$$

Since $\|\cdot\|_\lambda$ is upper semicontinuous, there exists a neighborhood W' of x, $W' \subset V \cap W$, such that

$$\|t_y - s_y\|_\lambda < C\delta + \|u - s_x\|_\lambda \quad (y \in W').$$

Clearly, $u \in \mathsf{T}(W', t, \mu, \delta)$. We claim that

$$\mathsf{T}(W', t, \mu, \delta) \subset \mathsf{T}(V, s, \lambda, \varepsilon). \tag{A.2}$$

Indeed, let $v \in \mathsf{T}(W', t, \mu, \delta)$, and let $y = p(v)$. We have

$$\begin{aligned} \|v - s_y\|_\lambda &\le \|v - t_y\|_\lambda + \|t_y - s_y\|_\lambda \\ &\le C\|v - t_y\|_\mu + \|t_y - s_y\|_\lambda < C\delta + C\delta + \|u - s_x\|_\lambda < \varepsilon. \end{aligned}$$

This implies (A.2) and completes the proof of (i). Part (ii) is immediate from (i). ☐

Lemma A.17. *Let X be a topological space, (E, p) be a prebundle of topological vector spaces over X, and $\mathscr{N} = \{\|\cdot\|_\lambda : \lambda \in \Lambda\}$ be a directed family of seminorms on E satisfying* (Ad1). *Then for each $x \in X$ the topology on E_x inherited from E coincides with the topology generated by the restrictions of the seminorms $\|\cdot\|_\lambda$ ($\lambda \in \Lambda$) to E_x.*

Proof. Let τ denote the topology on E_x inherited from E, and let τ' denote the topology on E_x generated by the restrictions of the seminorms $\|\cdot\|_\lambda$ ($\lambda \in \Lambda$). The standard base for τ' consists of all sets of the form

$$B_{\lambda,\varepsilon}(v) = \{u \in E_x : \|u - v\|_\lambda < \varepsilon\} \qquad (v \in E_x,\ \lambda \in \Lambda,\ \varepsilon > 0).$$

Since $\mathsf{T}(V, s, \lambda, \varepsilon) \cap E_x = B_{\lambda,\varepsilon}(s_x)$, (Ad1) implies that $\tau \subset \tau'$. On the other hand, we have

$$B_{\lambda,\varepsilon}(v) = v + B_{\lambda,\varepsilon}(0_x) = v + \mathsf{T}(X, 0, \lambda, \varepsilon) \cap E_x.$$

Together with Remark A.2, this implies that $B_{\lambda,\varepsilon}(v)$ is τ-open. Thus $\tau = \tau'$. □

Proposition A.18. *Let X be a topological space, and let (E, p) be a prebundle of topological vector spaces over X. A locally convex uniform vector structure $\mathscr{U}$ on E satisfies* (B0)–(B2) *if and only if $\mathscr{U} = \mathscr{U}_{\mathscr{N}}$ for an admissible directed family of seminorms on E. Specifically, given $\mathscr{U}$, we can take $\mathscr{N} = \{p_B : B \in \mathscr{B}\}$, where $\mathscr{B}$ is any base of $\mathscr{U}$ consisting of open sets.*

Proof. Let $\mathscr{N}$ be an admissible directed family of seminorms on E. The set $\mathsf{U}_{\lambda,\varepsilon} = \mathsf{T}(X, 0, \lambda, \varepsilon)$ is open by (Ad1), and so $\mathscr{U}_{\mathscr{N}}$ satisfies (B0). It is also immediate from (Ad1) and (A.1) that $\mathscr{U}_{\mathscr{N}}$ satisfies (B1). Finally, (B2) follows from (Ad2) and Lemma A.17.

Conversely, let $\mathscr{U}$ be a locally convex uniform vector structure on E satisfying (B0)–(B2), and let $\mathscr{B}$ be a base of $\mathscr{U}$ consisting of open sets. Applying Proposition A.7, we see that $\mathscr{U} = \mathscr{U}_{\mathscr{N}}$, where $\mathscr{N} = \{p_B : B \in \mathscr{B}\}$. We claim that $\mathscr{N}$ is admissible. Indeed, a standard argument (see, e.g., [142, I.4]) shows that for each $B \in \mathscr{B}$ we have $B = \mathsf{U}_{p_B,1}$. Together with (A.1), this implies that for each open set $V \subset X$, each $s \in \Gamma(V, E)$, and each $\varepsilon > 0$ we have

$$\mathsf{T}(V, s, p_B, \varepsilon) = s(V) + \mathsf{U}_{p_B,\varepsilon} = s(V) + \varepsilon \mathsf{U}_{p_B,1} = s(V) + \varepsilon B. \tag{A.3}$$

Now (Ad1) follows from (A.3) and Lemma A.12 (i). Finally, (Ad2) is immediate from (B2) and Lemma A.17. □

Remark A.19. According to Gierz [53], a locally convex bundle over X is a triple $(E, p, \mathscr{N})$, where (E, p) is a prebundle of topological vector spaces over X, and $\mathscr{N}$ is an admissible (in the strong sense, see Remark A.15) directed family of seminorms on E. For our purposes, however, the locally convex uniform vector structure determined by $\mathscr{N}$ is more important than $\mathscr{N}$ itself. Thus our point of view is closer to that of [35] and [170]. In some sense, the difference between our bundles and bundles in the sense of [53] is the same as between locally convex spaces and polynormed spaces (i.e., vector spaces endowed with distinguished families of seminorms, see [60]). However, in contrast to the case of topological vector spaces, neither the locally convex uniform vector structure determines the topology of E, nor vice versa.

Definition A.20. Let $(E, p, \mathscr{U})$ and $(E', p', \mathscr{U}')$ be locally convex bundles over X. A continuous map $f \colon E \to E'$ is a *bundle morphism* if the following holds:

(BM1) $p'f = p$;
(BM2) the restriction of f to each fiber E_x ($x \in X$) is a linear map from E_x to E'_x;
(BM3) for each $U' \in \mathscr{U}'$, we have $f^{-1}(U') \in \mathscr{U}$.

Remark A.21. Clearly, (iii) is equivalent to the uniform continuity of f with respect to the uniform structures $\tilde{\mathscr{U}}$ and $\tilde{\mathscr{U}}'$ (see Remark A.5).

The category of all locally convex bundles and bundle morphisms over X will be denoted by $\mathsf{Bnd}(X)$.

The following result (see [53, 5.2–5.7]) is a locally convex version of Fell's theorem [49] (cf. also [50, II.13.18], [177, C.25], [169], [170]). It provides a useful way of constructing locally convex bundles out of their fibers, seminorms, and sections.

Proposition A.22. *Let X be a topological space, (E,p) be a family of vector spaces over X, and $\mathscr{N} = \{\|\cdot\|_\lambda : \lambda \in \Lambda\}$ be a directed family of seminorms on E. Suppose that Γ is a vector subspace of $S(X,E)$ satisfying the following conditions:*

(S1) *For each $x \in X$, the set $\{s_x : s \in \Gamma\}$ is dense in E_x with respect to the topology generated by the restrictions of the seminorms $\|\cdot\|_\lambda$ $(\lambda \in \Lambda)$ to E_x;*
(S2) *For each $s \in \Gamma$ and each $\lambda \in \Lambda$, the map $X \to \mathbb{R}$, $x \mapsto \|s_x\|_\lambda$, is upper semicontinuous.*

Then there exists a unique topology on E such that $(E,p,\mathscr{U}_{\mathscr{N}})$ is a locally convex bundle and such that $\Gamma \subset \Gamma(X,E)$. Moreover, the family

$$\Big\{\mathsf{T}(V,s,\lambda,\varepsilon) : V \subset X \text{ open},\ s \in \Gamma,\ \lambda \in \Lambda,\ \varepsilon > 0\Big\} \tag{A.4}$$

is a base for the topology on E.

Remark A.23. The uniqueness part of Proposition A.22 is not proved in [53], so let us explain why the topology $\mathscr{T}$ with the above properties is unique, i.e., why (A.4) must be a base of $\mathscr{T}$. Let $W \subset E$ be an open set, and let $u \in W$. Without loss of generality, we may assume that $W = \mathsf{T}(V,s,\lambda,\varepsilon) \in \mathscr{B}(\mathscr{N})$. Let $x = p(u)$. Since $\|u - s_x\|_\lambda < \varepsilon$, we can choose $\delta > 0$ such that

$$2\delta + \|u - s_x\|_\lambda < \varepsilon.$$

By (S1), there exists $t \in \Gamma$ with $\|t_x - u\|_\lambda < \delta$. Then $\|t_x - s_x\|_\lambda < \delta + \|u - s_x\|_\lambda$. By the upper semicontinuity of $\|\cdot\|_\lambda$, there exists an open neighborhood V' of x such that $V' \subset V$ and

$$\|t_y - s_y\|_\lambda < \delta + \|u - s_x\|_\lambda \qquad (y \in V').$$

By construction, $u \in \mathsf{T}(V',t,\lambda,\delta)$. We claim that

$$\mathsf{T}(V',t,\lambda,\delta) \subset \mathsf{T}(V,s,\lambda,\varepsilon). \tag{A.5}$$

Indeed, let $v \in \mathsf{T}(V',t,\lambda,\delta)$, and let $y = p(v)$. We have

$$\|v - s_y\|_\lambda < \delta + \|t_y - s_y\|_\lambda < 2\delta + \|u - s_x\|_\lambda < \varepsilon.$$

This proves (A.5) and implies that (A.4) is a base of $\mathscr{T}$.

Alternatively, the uniqueness of $\mathscr{T}$ can be proved by adapting Fell's original argument (see [49], [50, II.13.18], or [177, C.25]) to the locally convex setting.

Remark A.24. Under the conditions of Proposition A.22, we can replace $\mathscr{N}$ by any directed subfamily equivalent to $\mathscr{N}$. By the uniqueness part of Proposition A.22, this will not affect the topology and the locally convex uniform vector structure on E.

Let us now describe a situation where conditions (S1) and (S2) of Proposition A.22 are satisfied automatically. Let K be a commutative algebra. By a *locally convex K-module* we mean a K-module M together with a locally convex topology such that for each $a \in K$ the map $M \to M$, $x \mapsto ax$, is continuous. The category of all locally convex K-modules and continuous K-module morphisms will be denoted by K-mod. Suppose that X is a topological space and $\gamma\colon K \to C(X)$ is an algebra homomorphism. Given

$a \in K$ and $x \in X$, we write $a(x)$ for $\gamma(a)(x)$. Define $\varepsilon_x \colon K \to \mathbb{C}$ by $\varepsilon_x(a) = a(x)$, and let $\mathfrak{m}_x = \operatorname{Ker} \varepsilon_x$. Given a locally convex K-module M and $u \in M$, let

$$M_x = M/\overline{\mathfrak{m}_x M}, \qquad u_x = u + \overline{\mathfrak{m}_x M} \in M_x. \tag{A.6}$$

We say that M_x is the *fiber* of M over $x \in X$. If $\|\cdot\|$ is a continuous seminorm on M, then the respective quotient seminorm on M_x will be denoted by the same symbol $\|\cdot\|$; this will not lead to confusion.

The following lemma is a locally convex version of [135, Proposition 1.2].

Lemma A.25. *For each $u \in M$, the function $X \to \mathbb{R}$, $x \mapsto \|u_x\|$, is upper semicontinuous.*

Proof. Let $x \in X$, and suppose that $\|u_x\| < C$. We need to show that $\|u_y\| < C$ as soon as y is close enough to x. We have

$$\inf\{\|u+v\| : v \in \mathfrak{m}_x M\} = \inf\{\|u+v\| : v \in \overline{\mathfrak{m}_x M}\} = \|u_x\| < C,$$

and so there exist $a_1, \ldots, a_n \in \mathfrak{m}_x$ and $v_1, \ldots, v_n \in M$ such that

$$\Bigl\| u + \sum_{i=1}^n a_i v_i \Bigr\| < C. \tag{A.7}$$

Observe that for each $a \in K$ and each $y \in X$ we have $a - a(y) \in \mathfrak{m}_y$. Hence

$$\|u_y\| \le \Bigl\| u + \sum_{i=1}^n (a_i - a_i(y)) v_i \Bigr\| \le \Bigl\| u + \sum_{i=1}^n a_i v_i \Bigr\| + \sum_{i=1}^n |a_i(y)| \|v_i\|. \tag{A.8}$$

Since each a_i is continuous and vanishes at x, (A.7) and (A.8) together imply that there exists a neighborhood V of x such that $\|u_y\| < C$ for all $y \in V$. This completes the proof. $\square$

We are now in a position to construct a fiber-preserving functor from $K\text{-}\mathsf{mod}$ to $\mathsf{Bnd}(X)$. Given a locally convex K-module M, let $\mathsf{E}(M) = \bigsqcup_{x\in X} M_x$, and let $p_M \colon \mathsf{E}(M) \to X$ be given by $p_M(M_x) = \{x\}$. Thus $(\mathsf{E}(M), p_M)$ is a family of vector spaces over X. Let $\mathscr{C}_M = \{\|\cdot\|_\lambda : \lambda \in \Lambda\}$ denote the family of all continuous seminorms on M. For each $\lambda \in \Lambda$ and each $x \in X$, the quotient seminorm of $\|\cdot\|_\lambda$ on M_x will be denoted by the same symbol $\|\cdot\|_\lambda$ (see discussion before Lemma A.25). Thus we obtain a directed family $\mathscr{N}_M = \{\|\cdot\|_\lambda : \lambda \in \Lambda\}$ of seminorms on $\mathsf{E}(M)$. The locally convex uniform vector structure on $\mathsf{E}(M)$ determined by $\mathscr{N}_M$ will be denoted by $\mathscr{U}_M$. For each $u \in M$, the function $\tilde{u} \colon X \to \mathsf{E}(M)$, $x \mapsto u_x$, is clearly a section of $(\mathsf{E}(M), p_M)$. Let $\Gamma_M = \{\tilde{u} : u \in M\}$. For each $x \in X$, we obviously have $\{\tilde{u}_x : \tilde{u} \in \Gamma_M\} = M_x$, and so (S1) holds. Lemma A.25 implies that (S2) holds as well. Applying Proposition A.22, we see that $(\mathsf{E}(M), p_M, \mathscr{U}_M)$ is a locally convex bundle over X. For brevity, we will denote every basic open set $\mathsf{T}(V, \tilde{u}, \lambda, \varepsilon)$ in $\mathsf{E}(M)$ (where $V \subset X$ is an open set, $u \in M$, $\lambda \in \Lambda$, and $\varepsilon > 0$) simply by $\mathsf{T}(V, u, \lambda, \varepsilon)$.

Remark A.26. In the above construction, we can let $\mathscr{C}_M$ be any directed defining family of seminorms on M. By Remark A.24, this will not affect the topology and the locally convex uniform vector structure on $\mathsf{E}(M)$.

Suppose now that $f \colon M \to N$ is a morphism in $K\text{-}\mathsf{mod}$. For each $x \in X$ we clearly have $f(\overline{\mathfrak{m}_x M}) \subset \overline{\mathfrak{m}_x N}$. Hence f induces a continuous linear map $f_x \colon M_x \to N_x$, $u_x \mapsto f(u)_x$. We let $\mathsf{E}(f) \colon \mathsf{E}(M) \to \mathsf{E}(N)$ denote the map whose restriction to each fiber M_x is f_x.

Lemma A.27. $\mathsf{E}(f)\colon \mathsf{E}(M)\to\mathsf{E}(N)$ *is a bundle morphism.*

Proof. Clearly, $\mathsf{E}(f)$ satisfies (BM1) and (BM2). Let $u\in M$ and $x\in X$, and let us prove the continuity of $\mathsf{E}(f)$ at $u_x\in M_x$. Let $\{\|\cdot\|_\lambda:\lambda\in\Lambda\}$ (respectively, $\{\|\cdot\|_\mu:\mu\in\Lambda'\}$) denote the family of all continuous seminorms on M (respectively, N). By Lemma A.12 (ii), a basic neighborhood of $\mathsf{E}(f)(u_x)=f(u)_x$ has the form $\mathsf{T}(V,f(u),\mu,\varepsilon)$, where $V\subset X$ is an open neighborhood of x, $\mu\in\Lambda'$, and $\varepsilon>0$. Since f is continuous, there exists $\lambda\in\Lambda$ such that for each $v\in M$ we have $\|f(v)\|_\mu=\|v\|_\lambda$. By passing to the quotients, we see that $\|f_y(v_y)\|_\mu\le\|v_y\|_\lambda$ ($v\in M$, $y\in X$). We claim that

$$\mathsf{E}(f)\big(\mathsf{T}(V,u,\lambda,\varepsilon)\big)\subset\mathsf{T}(V,f(u),\mu,\varepsilon). \tag{A.9}$$

Indeed, for each $v_y\in\mathsf{T}(V,u,\lambda,\varepsilon)$, where $v\in M$ and $y\in V$, we have

$$\|\mathsf{E}(f)(v_y)-f(u)_y\|_\mu=\|f_y(v_y-u_y)\|_\mu\le\|v_y-u_y\|_\lambda<\varepsilon.$$

This implies (A.9) and shows that $\mathsf{E}(f)$ is continuous. Finally, letting $u=0$ and $V=X$ in (A.9), we conclude that $\mathsf{E}(f)$ satisfies (BM3). $\square$

Summarizing, we obtain the following.

Theorem A.28. *There exists a functor* $\mathsf{E}\colon K\text{-}\mathsf{mod}\to\mathsf{Bnd}(X)$ *uniquely determined by the following properties:*

(i) *for each* $M\in K\text{-}\mathsf{mod}$ *and each* $x\in X$*, we have* $\mathsf{E}(M)_x=M_x$*;*
(ii) *the locally convex uniform vector structure on* $\mathsf{E}(M)$ *is determined by* $\mathscr{N}_M$*;*
(iii) *for each* $M\in K\text{-}\mathsf{mod}$ *and each* $u\in M$*, the section* $\tilde u\colon X\to\mathsf{E}(M)$*,* $x\mapsto u_x$*, is continuous;*
(iv) *for each morphism* $f\colon M\to N$ *in* $K\text{-}\mathsf{mod}$ *and each* $x\in X$*, we have* $\mathsf{E}(f)_x=f_x$*.*

For the purposes of Section 8, we need locally convex bundles with an additional algebraic structure. Let X be a set, and let (A,p) be a family of vector spaces over X. Suppose that each fiber A_x ($x\in X$) is endowed with an algebra structure, and let

$$\mathrm{mult}\colon A\times_X A\to A,\quad (u,v)\mapsto uv. \tag{A.10}$$

For each pair S,T of subsets of A, we let $S\cdot T=\mathrm{mult}((S\times T)\cap(A\times_X A))$.

Definition A.29. Let X be a topological space. By a *locally convex algebra bundle* over X we mean a locally convex bundle $(A,p,\mathscr{U})$ over X together with an algebra structure on each fiber A_x ($x\in X$) such that

(B3) the multiplication (A.10) is continuous;
(B4) for each $U\in\mathscr{U}$ there exists $V\in\mathscr{U}$ such that $V\cdot V\subset U$.

If, in addition, each fiber A_x is a Fréchet algebra with respect to the topology inherited from A, then we say that $(A,p,\mathscr{U})$ is a *Fréchet algebra bundle.* If $(A,p,\mathscr{U})$ and $(A',p',\mathscr{U}')$ are locally convex algebra bundles over X, then a bundle morphism $f\colon A\to A'$ is an *algebra bundle morphism* if the restriction of f to each fiber A_x ($x\in X$) is an algebra homomorphism from A_x to A'_x.

The category of all locally convex algebra bundles and algebra bundle morphisms over X will be denoted by $\mathsf{AlgBnd}(X)$.

We need the following modification of Proposition A.22.

Proposition A.30. *Under the conditions of Proposition* A.22*, suppose that each* E_x *is endowed with an algebra structure and that the following holds:*

(S3) *For each* $\lambda \in \Lambda$ *there exist* $\mu \in \Lambda$ *and* $C > 0$ *such that for each* $x \in X$ *and each* $u, v \in E_x$ *we have* $\|uv\|_\lambda \le C\|u\|_\mu \|v\|_\mu$*;*
(S4) Γ *is a subalgebra of* $S(X, E)$*.*

Then the bundle $(E, p, \mathscr{U}_{\mathscr{N}})$ *constructed in Proposition* A.22 *is a locally convex algebra bundle.*

Proof. Condition (S3) implies that for each $\varepsilon > 0$ we have $\mathsf{U}_{\mu,\delta} \cdot \mathsf{U}_{\mu,\delta} \subset \mathsf{U}_{\lambda,\varepsilon}$, where $\delta = \min\{\varepsilon/C, 1\}$. Thus (B4) holds. To prove (B3), let $u, v \in E \times_X E$, and let $\mathsf{T}(V, c, \lambda, \varepsilon)$ be a basic neighborhood of uv, where $V \subset X$ is an open set, $c \in \Gamma$, $\lambda \in \Lambda$, and $\varepsilon > 0$. Find $\mu \in \Lambda$ and $C > 0$ satisfying (S3). Fix any $\varepsilon' \in (0, \varepsilon)$ such that $uv \in \mathsf{T}(V, c, \lambda, \varepsilon')$, and choose $\delta > 0$ such that

$$C\delta(\|u\|_\mu + \|v\|_\mu + 5\delta) < \varepsilon - \varepsilon'. \tag{A.11}$$

Let $x = p(u) = p(v)$. By (S3), the multiplication on E_x is continuous, so (S1) implies that there exist $a, b \in \Gamma$ such that

$$\|u - a_x\|_\mu < \delta, \quad \|v - b_x\|_\mu < \delta, \tag{A.12}$$

$$a_x b_x \in \mathsf{T}(V, c, \lambda, \varepsilon'). \tag{A.13}$$

By (S4) and (A.13), there exists an open neighborhood W of x such that $W \subset V$ and

$$(ab)(W) \subset \mathsf{T}(V, c, \lambda, \varepsilon'). \tag{A.14}$$

By shrinking W if necessary and by using (S2), we can also assume that

$$\|a_y\|_\mu < \|a_x\|_\mu + \delta, \quad \|b_y\|_\mu < \|b_x\|_\mu + \delta \qquad (y \in W). \tag{A.15}$$

By (A.12), $\mathsf{T}(W, a, \mu, \delta)$ (respectively, $\mathsf{T}(W, b, \mu, \delta)$) is a neighborhood of u (respectively, v). We claim that

$$\mathsf{T}(W, a, \mu, \delta) \cdot \mathsf{T}(W, b, \mu, \delta) \subset \mathsf{T}(V, c, \lambda, \varepsilon). \tag{A.16}$$

Indeed, let $y \in W$, and let $u', v' \in E_y$ be such that $\|u' - a_y\|_\mu < \delta$ and $\|v' - b_y\|_\mu < \delta$. We have

$$\begin{aligned}
\|u'v' - c_y\|_\lambda &\le \|u'v' - a_y b_y\|_\lambda + \|a_y b_y - c_y\|_\lambda \\
&\le \|(u' - a_y)v'\|_\lambda + \|a_y(v' - b_y)\|_\lambda + \varepsilon' && \text{(by (A.14))} \\
&\le C\|u' - a_y\|_\mu \|v'\|_\mu + C\|a_y\|_\mu \|v' - b_y\|_\mu + \varepsilon' && \text{(by (S3))} \\
&< C\delta(\|a_y\|_\mu + \|v'\|_\mu) + \varepsilon' \\
&< C\delta(\|a_y\|_\mu + \|b_y\|_\mu + \delta) + \varepsilon' \\
&< C\delta(\|a_x\|_\mu + \|b_x\|_\mu + 3\delta) + \varepsilon' && \text{(by (A.15))} \\
&< C\delta(\|u\|_\mu + \|v\|_\mu + 5\delta) + \varepsilon' && \text{(by (A.12))} \\
&< (\varepsilon - \varepsilon') + \varepsilon' = \varepsilon && \text{(by (A.11)).}
\end{aligned}$$

Thus (A.16) holds, which implies (B3) and completes the proof. $\square$

Let K be a commutative algebra. By a *locally convex* K*-algebra* we mean a locally convex K-module A together with a continuous K-bilinear multiplication $A \times A \to A$. Morphisms of locally convex K-algebras are defined in the obvious way. The category of locally convex K-algebras will be denoted by $K\text{-}\mathsf{alg}$. As above, let X be a topological space, and let $\gamma \colon K \to C(X)$ be an algebra homomorphism. Observe that for each locally convex K-algebra A and each $x \in X$ the subspace $\overline{\mathfrak{m}_x A}$ is a two-sided ideal of A. Thus

the fiber $A_x = A/\overline{\mathfrak{m}_x A}$ of A over x is a locally convex algebra in a natural way. Let $\mathscr{C}_A = \{\|\cdot\|_\lambda : \lambda \in \Lambda\}$ denote the family of all continuous seminorms on A. Since the multiplication on A is continuous, it follows that for each $\lambda \in \Lambda$ there exist $\mu \in \Lambda$ and $C > 0$ such that for all $a, b \in A$ we have $\|ab\|_\lambda \le C\|a\|_\mu \|b\|_\mu$. By passing to the quotient seminorms on the fibers A_x ($x \in X$), we see that (S3) holds (with $E_x = A_x$). Clearly, Γ_A satisfies (S4). Applying Proposition A.30, we conclude that $\mathsf{E}(A)$ is a locally convex algebra bundle. Moreover, if $f \colon A \to B$ is a locally convex K-algebra morphism, then $\mathsf{E}(f) \colon \mathsf{E}(A) \to \mathsf{E}(B)$ is an algebra bundle morphism. Thus we have the following analog of Theorem A.28.

Theorem A.31. *There exists a functor* $\mathsf{E} \colon K\text{-}\mathsf{alg} \to \mathsf{AlgBnd}(X)$ *uniquely determined by the following properties:*

(i) *for each* $A \in K\text{-}\mathsf{alg}$ *and each* $x \in X$*, we have* $\mathsf{E}(A)_x = A_x$*;*
(ii) *the locally convex uniform vector structure on* $\mathsf{E}(A)$ *is determined by* $\mathscr{N}_A$*;*
(iii) *for each* $A \in K\text{-}\mathsf{alg}$ *and each* $u \in A$*, the section* $\tilde{u} \colon X \to \mathsf{E}(A)$*,* $x \mapsto u_x$*, is continuous;*
(iv) *for each morphism* $f \colon A \to B$ *in* $K\text{-}\mathsf{alg}$ *and each* $x \in X$*, we have* $\mathsf{E}(f)_x = f_x$*.*

In conclusion, let us discuss the notion of continuity for locally convex bundles. Let $(E, p, \mathscr{U})$ be a locally convex bundle over a topological space X. Recall that we always have $\mathscr{U} = \mathscr{U}_{\mathscr{N}}$, where $\mathscr{N}$ is an admissible directed family of seminorms on E (see Proposition A.18). By Remark A.14, each seminorm belonging to $\mathscr{N}$ is upper semicontinuous. In the theory of Banach bundles (see, e.g., [50]), it is customary to consider only those Banach bundles whose norm is a continuous function on E. This leads naturally to the following definition.

Definition A.32. We say that a locally convex bundle $(E, p, \mathscr{U})$ is *continuous* if there exists an admissible directed family $\mathscr{N}$ of continuous seminorms on E such that $\mathscr{U} = \mathscr{U}_{\mathscr{N}}$.

The following result gives a convenient way of proving the continuity of locally convex bundles.

Proposition A.33. *Let* $(E, p, \mathscr{U})$ *be a locally convex bundle over a topological space* X*, and let* Γ *be a vector subspace of* $\Gamma(X, E)$ *such that for each* $x \in X$ *the set* $\{s_x : s \in \Gamma\}$ *is dense in* E_x*. Suppose that* $\|\cdot\|$ *is an upper semicontinuous seminorm on* E *such that for every* $s \in \Gamma$ *the map* $X \to \mathbb{R}$*,* $x \mapsto \|s_x\|$*, is continuous. Then* $\|\cdot\|$ *is continuous.*

Proof (cf. [177], the last step of the proof of C.25). We have to show that $\|\cdot\|$ is lower semicontinuous. Let $u \in E$, let $x = p(u)$, and suppose that $\|u\| > c > 0$. Choose $\delta > 0$ such that $\|u\| > c + 2\delta$. Since $\|\cdot\|$ is upper semicontinuous and the topology on E_x is translation invariant, it follows that $\{v \in E_x : \|v - u\| < \delta\}$ is open in E_x. Hence there exists $s \in \Gamma$ such that $\|u - s_x\| < \delta$. In particular, $\|s_x\| > c + \delta$. By assumption, this implies that there exists an open neighborhood V of x such that $\|s_y\| > c+\delta$ for all $y \in V$. By Remark A.14, $\mathsf{T}(V, s, \|\cdot\|, \delta)$ is an open neighborhood of u. If now $v \in \mathsf{T}(V, s, \|\cdot\|, \delta)$ and $y = p(v)$, then

$$\|v\| \ge \|s_y\| - \|s_y - v\| > c + \delta - \delta = c.$$

This implies that $\|\cdot\|$ is lower semicontinuous and completes the proof. $\square$

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